Complex Numbers

Spring 2019

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  - x is the **real part** of z ($\text{Re}(z)$)
  - y is the **Imaginary part** of z ($\text{Im}(z)$)
- i is the **Imaginary unit** defined by the property

$$\boldsymbol{i}^2 = -1$$

Why Complex Numbers?

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The **field** (a set on which $+, -, \times, /$ are defined) of real numbers is not closed **algebraically**, i.e. there exist polynomials with real coefficients but do not have any real solutions. For example

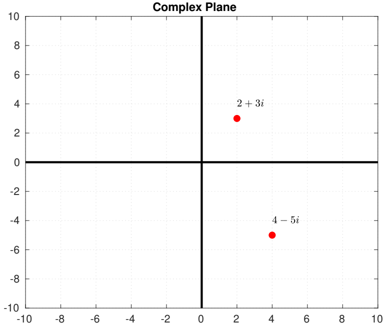
$$\boxed{x^2 = -2}$$

has no roots in $\mathbb{R}$. However, for $x \in \mathbb{C}$ using the definition of $\boldsymbol{i}$, we note that

$$\boxed{-1 = \boldsymbol{i}^2 \iff x^2 = (-1)(2) = 2\boldsymbol{i}^2 \implies x = \pm\sqrt{2}\boldsymbol{i}}$$

Complex plane

- $z = x + \boldsymbol{i}y \in \mathbb{C}$ has two independent components (**real part** x and **imaginary part** y).
- As a result a 2D plane is needed to represent all possible combinations of x and y.
- The x-axis corresponds to the **real axis** and y-axis is the **imaginary** axis.



Representation of $2 + 3\boldsymbol{i}$ and $4 - 5\boldsymbol{i}$

Working with complex numbers

Let $z_1 = x_1 + \boldsymbol{i} y_1$ and $z_2 = x_2 + \boldsymbol{i} y_2$

addition

$$z_1 + z_2 = (x_1 + x_2) + \boldsymbol{i}(y_1 + y_2)$$

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multiplication

$$\begin{aligned} z_1 z_2 &= (x_1 + \boldsymbol{i} y_1)(x_2 + \boldsymbol{i} y_2) \\ &= x_1 x_2 + \boldsymbol{i} x_1 y_2 + \boldsymbol{i} y_1 x_2 + \boldsymbol{i}^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + \boldsymbol{i}(x_1 y_2 + y_1 x_2) \end{aligned}$$

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Let $z_1 = x_1 + \boldsymbol{i} y_1$ and $z_2 = x_2 + \boldsymbol{i} y_2$

Complex conjugate

$$\bar{z} := x - \boldsymbol{i} y \text{ is the } \textbf{complex conjugate} \text{ of } z = x + \boldsymbol{i} y$$

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Computing Im and Re parts using complex conjugate

$$\begin{aligned} \mathrm{Re}(z) &= \frac{z + \bar{z}}{2} = \frac{(x + \boldsymbol{i} y) + (x - \boldsymbol{i} y)}{2} = x \\ \mathrm{Im}(z) &= \frac{z - \bar{z}}{2\boldsymbol{i}} = \frac{(x + \boldsymbol{i} y) - (x - \boldsymbol{i} y)}{2\boldsymbol{i}} = y \end{aligned}$$

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absolute value

$$|z| := \sqrt{x^2 + y^2} = \sqrt{(x + \boldsymbol{i} y)(x - \boldsymbol{i} y)} = \sqrt{z\bar{z}}$$

Working with complex numbers

division

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{x_1 + \boldsymbol{i} y_1}{x_2 + \boldsymbol{i} y_2} \\ &= \left(\frac{x_1 + \boldsymbol{i} y_1}{x_2 + \boldsymbol{i} y_2}\right)\left(\frac{x_2 - \boldsymbol{i} y_2}{x_2 - \boldsymbol{i} y_2}\right) \text{ (make the denominator real)} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \boldsymbol{i}\frac{x_1 x_2 - y_1 x_2}{x_2^2 + y_2^2}\end{aligned}$$

Complex numbers in MATLAB

WARNING: Do not use the ***i*** as a variable in your code.

- Defining complex numbers: `>> z1=2+3i; z2 = 4-5i;` or `>>z1 = complex(2,3)` (**Use this option, especially if you want to plot real numbers on the complex plane**)
- To extract the **real** and **imaginary parts** use the MATLAB functions `real` and `imag`, resp. as
- Use `norm` and `conj` to compute $|z|$ and $\bar{z}$, resp.

```
>> z1=2+3i; z2 = 4-5i;
>> real(z1)
ans =
      2
>> imag(z1)
ans =
      3
```

```
>> z1=2+3i; z2 = 4-5i;
>> norm(z1)
ans =
     3.6056
>> conj(z1)
ans =
    2.0000 - 3.0000i
```

We can also define functions and do complex arithmetic as usual

Complex numbers in MATLAB - plotting

Plotting points

Use the MATLAB `plot` function as `plot(z,LineSpec)`. e.g. to plot a red dotted complex point of size 20:

```
>>plot(z1,'r.','MarkerSize',20)
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Plotting lines

Again use the MATLAB `plot` function e.g.

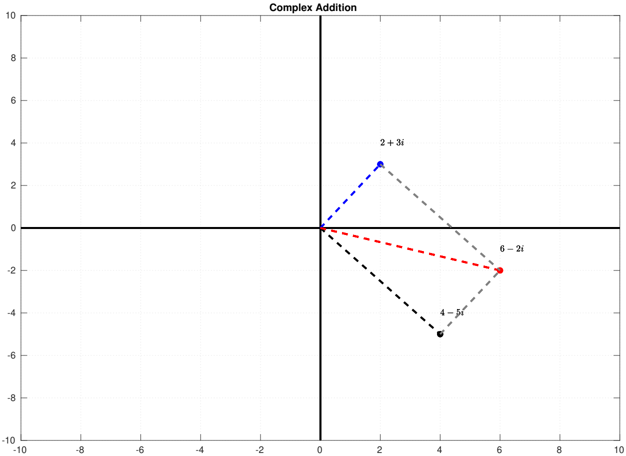
```
>>plot([z0 z1],'b--','Linewidth',2)
```

will join the points $z1$ and $z2$ with a black dashed line.

Adding complex numbers - a geometric view

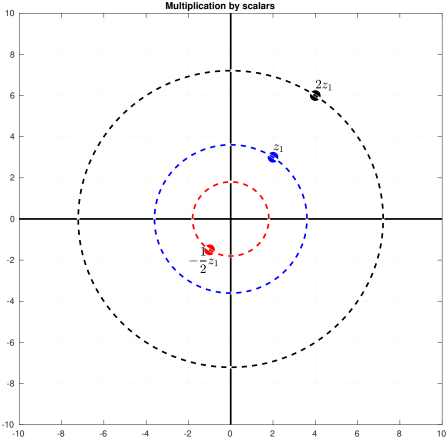
Parallelogram law

$$z1 = 2 + 3\boldsymbol{i} \quad z2 = 4 - 5\boldsymbol{i} \quad z3 = z1 + z2 = 6 - 2\boldsymbol{i}$$



Multiplication by scalars

$$\boxed{z_1 = 2 + 3\boldsymbol{i}}$$



Multiplying complex numbers

just "foil" it out

If $z_1 = x_1 + \boldsymbol{i} y_1$ and $z_2 = x_2 + \boldsymbol{i} y_2$

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BUT...to really appreciate this let's doing some plotting

Multiplying complex numbers – Polar coordinates

Recall that given a point (x, y) in $\mathbb{R}^2$, we can write this point in the form (r, θ) with

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2 + y^2 &= r^2 \\ \frac{y}{x} &= \tan\theta \end{aligned}}$$

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Polar Representation of Complex numbers

If $z = x + \boldsymbol{i}y$ then we can write z as:

$$\boxed{\begin{aligned} z &= r\cos\theta + \boldsymbol{i}r\sin\theta \\ r &= |z| = \sqrt{x^2 + y^2} \end{aligned}}$$

de Moivre's Formula

When $z = r\cos\theta + \boldsymbol{i}r\sin\theta$, and n is any natural number,

$$\boxed{z^n = r^n\cos(n\theta) + \boldsymbol{i}r^n\sin(n\theta)}$$

- This means when we compute z^n the result is a complex number with length raised to the power n and rotated by an angle $n\theta$.

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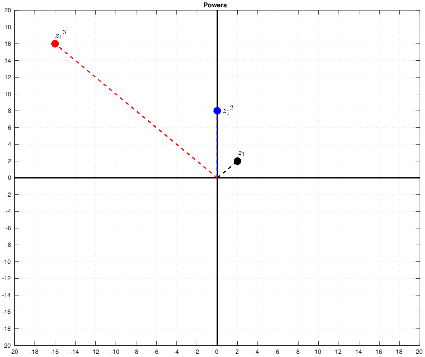


$$z_1 = 2 + 2\boldsymbol{i} \rightarrow |z| = 2\sqrt{2}, \theta = \frac{\pi}{4}$$
$$z_1^2 \text{ has } r^2 = (2\sqrt{2})^2 = 8, 2\theta = \frac{\pi}{2}$$
$$z_1^3 \text{ has } r^3 = 8\sqrt{8} \text{ and } 3\theta = \frac{3\pi}{4}$$

Multiplying complex numbers

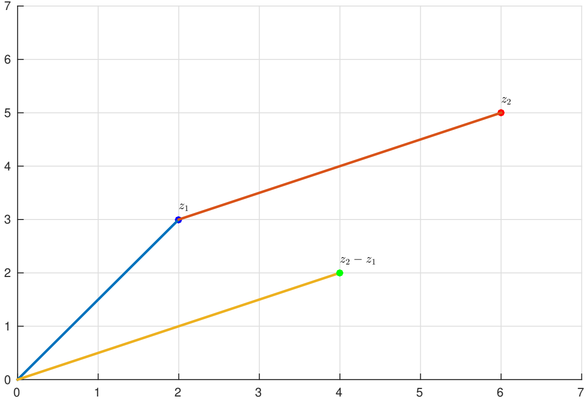
If $z_1 = r\cos(\theta) + \boldsymbol{i}\sin(\theta)$ and $z_2 = s\cos(\psi) + \boldsymbol{i}s\sin(\psi)$, one can show (using trig identities) that

$$z_1 z_2 = rs\cos(\theta + \psi) + \boldsymbol{i}rs\sin(\theta + \psi)$$

lengths are multiplied and angle arguments are added

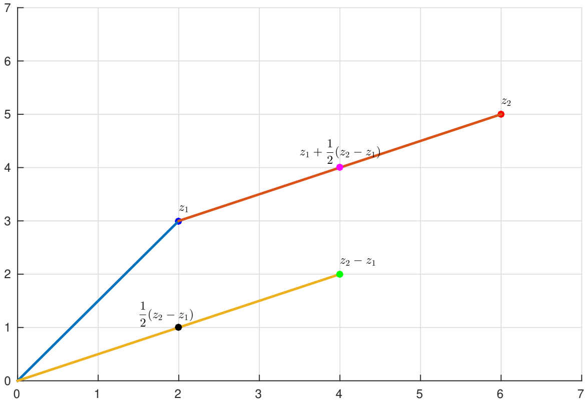
Segments in the complex plane

$$z_1 = 2 + 3\boldsymbol{i} \quad z_2 = 6 + 5\boldsymbol{i}$$

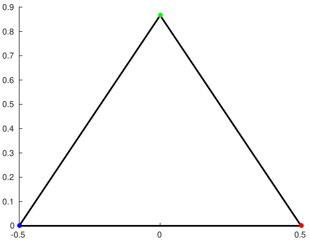


Segments in the complex plane

$$z_1 = 2 + 3\boldsymbol{i} \quad z_2 = 6 + 5\boldsymbol{i}, \quad \frac{1}{2}(z_2 - z_1) \quad z_1 + \frac{1}{2}(z_2 - z_1)$$



Chaos game



Rules

1. Color each vertex of an equilateral triangle with a different color.
2. Color a six-sided die so that 2 faces are red, 2 are yellow and 2 are blue
3. Choose a random starting point inside the triangle (this rule may be relaxed)
4. Roll the die.
5. Move half the distance from the seed towards the vertex with the same color as the number rolled.
6. Roll again from the point marked, move half the distance towards the vertex of the same color as the number rolled.
7. Mark the point, repeat.

Chaos game

Generalize the `chaos.m` script to a 5 sided die and a regular pentagon with coordinates

$$0 + \boldsymbol{i}$$

$$-\frac{1}{4}\sqrt{10 + 2\sqrt{5}} + \frac{1}{4}(\sqrt{5} - 1)\boldsymbol{i}$$

$$-\frac{1}{4}\sqrt{10 - 2\sqrt{5}} - \frac{1}{4}(\sqrt{5 + 1})\boldsymbol{i}$$

$$\frac{1}{4}\sqrt{10 - 2\sqrt{5}} - \frac{1}{4}(\sqrt{5 + 1})\boldsymbol{i}$$

$$-\frac{1}{4}\sqrt{10 + 2\sqrt{5}} + \frac{1}{4}(\sqrt{5 - 1})\boldsymbol{i}$$