

Linear Transformations

Spring 2019

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Linear transformations preserve lines, unlike nonlinear transformations that may transform a line segment into a parabolic curve, or ellipse

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- A is a 2×2 matrix and \vec{v} is a 2×1 column vector.
- Special examples of linear transformations include:
 - 1 scaling transformations
 - 2 rotations
 - 3 translations

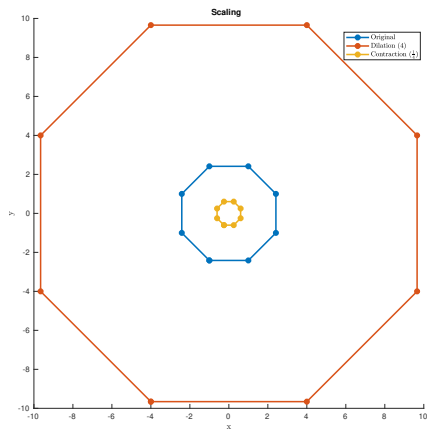
Scaling Transformations

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(\vec{v}) = c\vec{v}$ for $c \in (0, \infty)$

- $c > 1$ - dilation by a factor of c
- $c < 1$ - contraction by a factor of c
- In matrix form

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

Scaling Transformations



Rotations

Rotations by an angle θ about the origin where the rotation is measured from the positive x -axis in an anticlockwise direction

- In matrix form, the linear transformation can be represented as:

$$\mathbf{T} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflections

Reflections about a line L through the origin, e.g.

- Reflecting a point in \mathbb{R}^2 about the y -axis:

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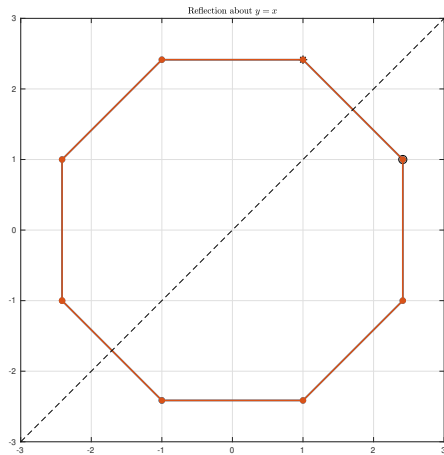
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- In general, the transformation corresponding to a reflection about the line L making an angle θ with the positive x -axis is given by

$$\mathbf{A} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad a^2 + b^2 = 1$$

Reflection



Shear

- y -shear

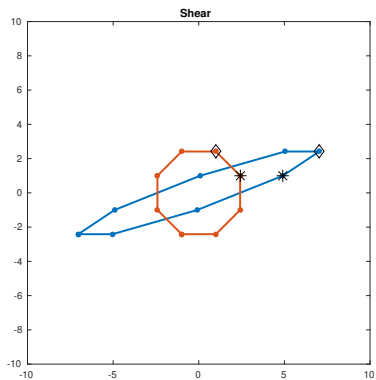
$$T = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

- x -shear

$$T = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

x-shear

$$T = \begin{bmatrix} 1 & 2.5 \\ 0 & 1 \end{bmatrix}$$



Compositions of transformations

Given two linear transformations \mathbf{T} and \mathbf{S} both $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ with

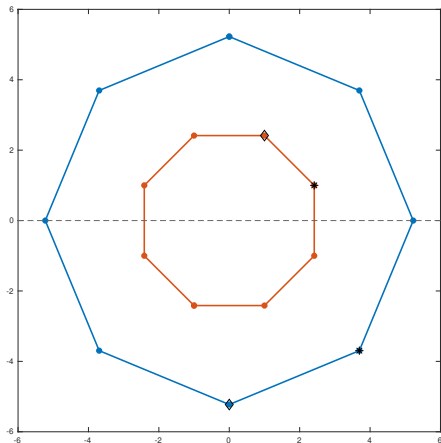
$$\mathbf{T}(\vec{v}) = \mathbf{A}\vec{v} \text{ and } \mathbf{S}(\vec{v}) = \mathbf{B}\vec{v} \quad \forall \vec{v} \in \mathbb{R}^2$$

then the composition of the transformation \mathbf{T} and \mathbf{S} , $\mathbf{T} \circ \mathbf{S}$ **AB**

$$(\mathbf{T} \circ \mathbf{S})(\vec{v}) = \mathbf{T}(\mathbf{S}(\vec{v})) = \mathbf{T}(\mathbf{B}\vec{v}) = \mathbf{A}\mathbf{B}\vec{v}$$

Compositions of transformations

Rotation $\theta = \frac{\pi}{8}$ then reflection about $y = 0$, then dilation by a factor of 2.



Orthogonal transformations

- A linear transformation $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called **orthogonal** if it preserves the length of vectors:

$$\|\mathbf{T}(\vec{v})\| = \|\vec{v}\|, \quad \forall \vec{v} \in \mathbb{R}^n$$

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- 1 $\|\mathbf{A}\vec{v}\| = \|\vec{v}\|, \quad \forall \vec{v} \in \mathbb{R}^n$
- 2 The columns of \mathbf{A} form an orthonormal basis of \mathbb{R}^n
- 3 $\mathbf{A}^T \mathbf{A} = \mathbf{I}_n$
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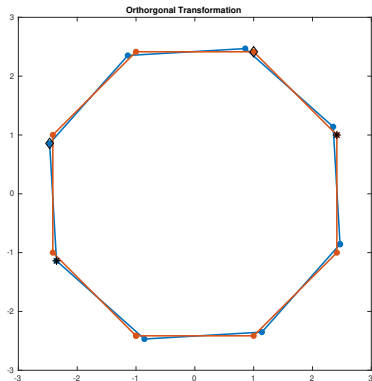
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- Orthogonal transformations also preserve dot products of vectors and thus angles are preserved

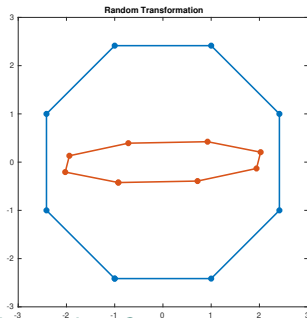
Random Orthogonal transformations

```
T= orth(rand(2,2))
```



Random Transformation

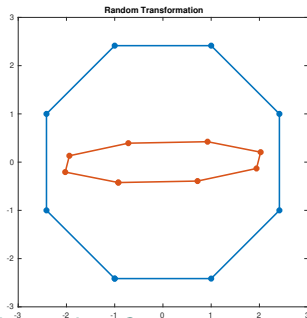
$$M = \begin{bmatrix} 0.8212 & 0.0430 \\ 0.0154 & 0.1690 \end{bmatrix}$$



Can this transformation be undone?

Random Transformation

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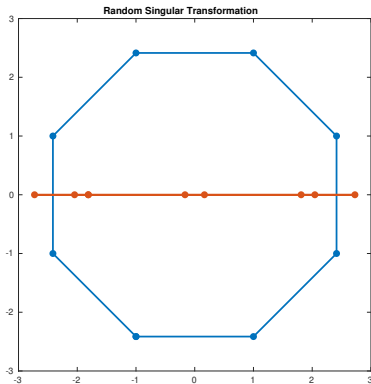


Can this transformation be undone?

Yes! $\det(M) = 0.1381$

Random non-invertible Transformation

$$M = \begin{bmatrix} 0.9884 & 0.3409 \\ 0.0000 & 0.0000 \end{bmatrix}$$



Affine transformations

These are mappings of the form

$$T(\vec{v}) = \mathbf{A}\vec{v} + \vec{b}$$

i.e. **affine transformations** are composed of a linear transformation ($\mathbf{A}\vec{v}$) then shifted in the direction \vec{b}

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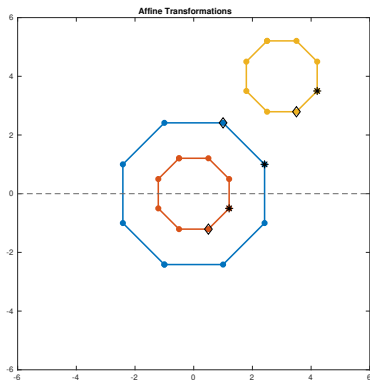
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i.e. **affine transformations** are composed of a linear transformation ($\mathbf{A}\vec{v}$) then shifted in the direction \vec{b}

- Affine transformations preserve collinearity and ratios of distances.
- Translations, dilations, contractions, reflections and rotations are all examples of affine transformations.

Affine transformations

$$\mathbf{T} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



Affine transformations and fractals

Consider **three** different linear transformations on points (x, y) starting at $(0, 0)$ and **one** linear transformation performed randomly with different probabilities

- 85% of the time:

$$T_1 = \mathbf{A}_1 \vec{v} + \vec{b}_1 = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \vec{v} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

- 7% of the time:

$$T_2 = \mathbf{A}_2 \vec{v} + \vec{b}_2 = \begin{bmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \vec{v} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

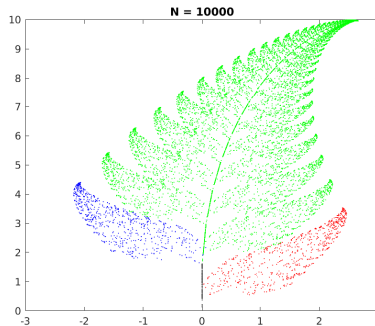
- 7% of the time:

$$T_3 = \mathbf{A}_3 \vec{v} + \vec{b}_3 = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \vec{v} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}$$

- 1% of the time:

$$T_4 = \mathbf{A}_4 \vec{v} = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \vec{v}$$

Affine transformations and fractals



Affine transformations and fractals

- `randsample(4,1,true,[0.85 0.07 0.07 0.01])` to generate random integers with weights
- Preallocate a matrix to store the computed points
- Store each computed point in a matrix, then plot the matrix