Newton's method

Spring 2019

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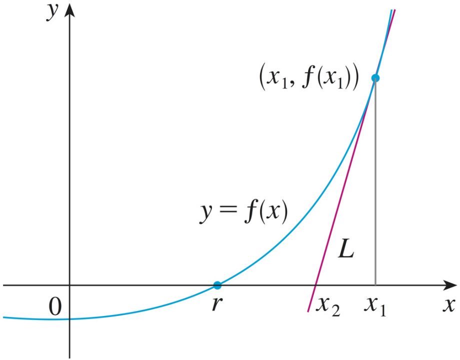
- Starting with an initial guess x_1, approximate $f(x)$ by the tangent line, L and use that to obtain a new approximate, x_2,



A formula for the approximations

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$$y - f(x_1) = f'(x_1)(x - x_1)$$

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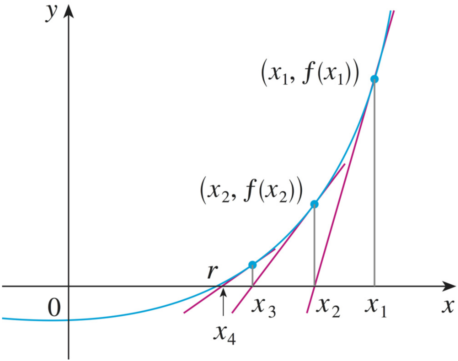
x_2 is our second approximation and is closer to the root, r.

Repeat!



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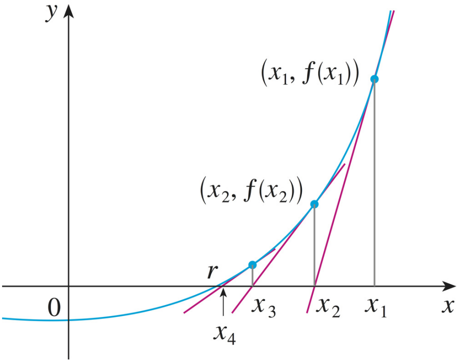

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Convergence

- We obtain a sequence of approximations $x_1, x_2, x3, \ldots$, where

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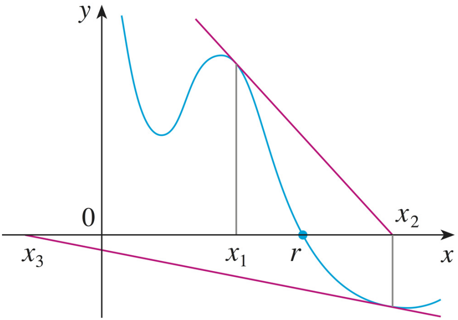
- We can quantify how fast the convergence occurs (see MA 427).

Newton's method may fail



The initial guess needs to be sufficiently close to r

Implementing Newton's method: Stopping criterion

We Loop until we are satisfied by the approximation x_{n+1} to r. In most practical cases the true solution is not known so $|x_{n+1} - r|$ cannot be computed, so we approximate the error:

$$\boxed{|x_{n+1} - x_n| \approx |x_{n+1} - r|}$$

Given an error tolerance, ϵ, stop the loop when

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Other stopping criteria:

- $|f(x_{n+1})| \leq \epsilon$ or
- $\left|\frac{x_{n+1} - x_n}{x_n}\right| \leq \epsilon$

Implementing Newton's method

```
>>[r, its] = newton_solver(f,x1,espilon)
```

Input: initial guess and f
Output: root r and number of iterations,its
while $((|x_{n+1} - x_n| > \epsilon)$ *and* $(its < MAX\_ITS))$ **do**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$its = its + 1$

end

Algorithm 1: Newton's method

Use the MATLAB Symbolic Package to find and evaluate the derivative of `f`.