12 LINEAR FUNCTIONS

Functions whose graphs are straight lines

\[ \text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = m \]

\[ y = mx + b \text{, where } b \text{ is the } y \text{ intercept.} \]

* The slope, \( m \), is the rate of change of \( y \) with respect to \( x \).

**Back to our cost function**

<table>
<thead>
<tr>
<th>( q )</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(q) )</td>
<td>500</td>
<td>550</td>
<td>600</td>
<td>650</td>
<td>700</td>
</tr>
</tbody>
</table>

**Graph**

\[ C(q) \text{ is a linear function} \]

\[ \text{Slope} = \frac{\text{change in cost}}{\text{change in } q} = \frac{600 - 550}{70 - 60} = \frac{50}{10} = \$5/\text{item} \]

The slope in this case is the cost of producing one additional item. (Constant Rate of Increase)

**General linear functions** \( (y = mx + b) \), where \( m \) = slope and \( b \) = \( y \) intercept.

**Suppose**

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>25</td>
<td>30</td>
<td>35</td>
<td>40</td>
</tr>
</tbody>
</table>

From the table, how can we guess that this is a linear function?
The formula is $y = 5x + 25$.

Equation of a line given 2 points: $y - y_0 = m(x - x_0)$.

Example:

Equation of line with slope 2 passing through (1,2):

$y - 2 = 2(x - 1)$

$y = 2 + 2x - 2$

$y = 2x$.
Equation of line given 2 points

Equation of line passing through \((0,2)\) and \((2,3)\)

\[
m = \frac{\Delta y}{\Delta x} = \frac{3 - 2}{2 - 0} = \frac{1}{2}
\]

We can then use the point-slope form

\[
y - y_1 = m(x - x_1)
\]

\[
y - 3 = \frac{1}{2}(x - 2)
\]

or

\[
y = 3 + \frac{x - 1}{2}
\]

\[
y = \frac{3x}{2} + 5
\]

Determining slope and \(y\)-intercept from equation

Given \(7y + 12x - 2 = 0\)

Convert into the form \(y = mx + b\)

\[
7y + 12x - 2 = 0
\]

\[
7y = 2 - 12x
\]

\[
y = \frac{2}{7} - \frac{12}{7}x
\]

\[
\Rightarrow \quad \text{Intercept} = \frac{2}{7}
\]

\[
\text{Slope} = -\frac{12}{7}
\]

Example

A city's population was 30,000 in the year 2015 and it growing by 850 people per year.

(a) Give a formula for the city's population \(P\) as a function of \(t\), the number of years since 2010.

(b) What is the predicted population in 2020?

(c) When is the population expected to reach 45,000?
(a) \[ P(t) = 30,700 + 850t \]

(b) Year 2010 is \( t = 0 \), Year 2020, \( t = 10 \).
\[ P(10) = 30,700 + 850 \times 10 = 39,200. \]

(c) \[ \text{Find } t \text{ such that} \]
\[ 45,000 = 30,700 + 850t \]
\[ 45,000 - 30,700 = 850t \]
\[ \Rightarrow t = \frac{14,300}{850} \Rightarrow t = 16.82 \]

So in the year 2026.

Example (They do) Type out Section 1.2

The annual revenue for McDonald's can be estimated by
\[ R(t) = 19.1 + 1.8t \]

where \( R \) is the revenue in billions and \( t \) is time since 2005.

(a) What is the slope of \( R \) (include units) (interpret in terms of Revenue)

Slope = 1.8 billion dollars/year.

The revenue is increasing at a rate of 1.8 billion dollars/year.

(b) What is the vertical intercept and interpret

19.1 billion dollars.

The revenue in 2005 was $19.1 billion.
Families of Linear Functions

1. Lines with 0 intercept
   - Decreasing rate of change: $y = -2x$
   - Increasing rate of change: $y = 2x$
   - $y = x$
   - $y = \frac{1}{2}x$

2. Constant slope
   - $y = x + 1$
   - $y = x$
   - $y = x - 1$

Summary

Linear functions describe a constant rate of change.

Revenue is increasing at 5% per year.

$y = mx + b$. 

Reasoning about the diagrams:
- **Decreasing Rate of Change** (Red lines): $y = -2x$, $y = -\frac{1}{2}x$
- **Increasing Rate of Change** (Blue lines): $y = 2x$, $y = \frac{1}{2}x$
- **Constant Slope** (Black lines): $y = x + 1$, $y = x$, $y = x - 1$