1.5 Exponential Functions

\[ f(x) = 2^x \]  
\[ \text{the power is a variable } x. \]

In general, exponential functions take the form \( f(x) = k a^x \).

\[ f(0) = k a^0 = k. \] i.e \( a^0 = 1 \) for any constant \( a \).

Exponential functions are ideal for describing population growth.

<table>
<thead>
<tr>
<th>YEAR</th>
<th>Population (Millions)</th>
<th>Change in population</th>
<th>Ratio of pop</th>
</tr>
</thead>
<tbody>
<tr>
<td>2003</td>
<td>12.853</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2004</td>
<td>12.290</td>
<td>0.487</td>
<td>1.024</td>
</tr>
<tr>
<td>2005</td>
<td>12.747</td>
<td>0.457</td>
<td>1.024</td>
</tr>
<tr>
<td>2006</td>
<td>14.225</td>
<td>0.478</td>
<td>1.034</td>
</tr>
</tbody>
</table>

\[ \frac{13.290}{12.853} \]

Remarks:

1. The rate of growth of the population is not constant \( \Rightarrow \) the population growth is NOT linear.

2. The ratio of population in successive years is constant.

If we let \( t \) be the time in years since 2003, then

\[ P(0) = 12.853 \]

\[ \frac{P(1)}{P(0)} = 1.024 \] \( \Rightarrow \) \( P(1) = P(0) \cdot 1.024 \)

\[ \frac{P(2)}{P(1)} = 1.024 \] \( \Rightarrow \) \( P(2) = P(1) \cdot 1.024 \) but \( P(1) = P(0) \cdot 1.024 \)

\[ \therefore \] \( P(2) = P(0) \cdot (1.024)^2 \)

\( \vdots \)

\[ P(t) = 12.853 \cdot (1.024)^t \] million. \( \text{[The population is growing at } 3.4\% \text{ per year]} \).
\[ P(t) = 12.853 \cdot (1.034)^t \]

*Initial population*

*Growth factor (amt by which pop grows each year)*

\[ P(t) = 12.853(1.034)^t \]

*(concave up)*

- The population is growing faster as time passes.
- In general, exponential functions increase slowly at first then rise quickly.

**Exponential Decay**

Suppose we start with an initial dose of 250mg of drug in the blood.

Suppose the concentration of drug decreases by 46% per hour.

Let \( Q(t) \) be the concentration of drug at time \( t \).

\[
\begin{align*}
Q(0) &= 250\text{mg} \\
Q(1) &= 250 \cdot (1 - 0.4) = 250 \cdot 0.6 \\
Q(2) &= Q(1) \cdot 0.6 = 250 \cdot (0.6)^2 \\
Q(3) &= Q(2) \cdot 0.6 = 250 \cdot (0.6)^3 \\
\end{align*}
\]

So after \( t \) hours, the concentration of drug is

\[ Q(t) = 250 \cdot (0.6)^t \]

This is called exponential decay.
The graph is decreasing and concave up.

In general exponential functions take the form

$$ P = P_0 a^t $$

where

1. $P_0$ is the initial amount
2. $a$ is the factor by which $P$ changes when $t$ increases by 1
   - $a > 1$ means we have exponential growth
   - $0 < a < 1$ means we have exponential decay
3. $a = 1 + r$, where $r$ is a percentage rate of change

In our examples:

1. $P(t) = P_0 (1.034)^t \Rightarrow a = 1.034 = 1 + 0.034 \Rightarrow r = 3.4\%$
2. $Q(t) = Q_0 (0.6)^t \Rightarrow a = 0.6 = 1 - 0.4 \Rightarrow r = 40\%$. 
1. The amount of drug in a body is 15mg. Find a formula for $A$, the amount in mg, at a time $t$ minutes later if $A$ is
(a) Decreasing by 0.4mg per min (Constant rate of change)
(b) Decreasing by 3% per minute. (Constant percentage change)

(a) $A(t) = 15 - 0.4t$

(b) $A(t) = 15(1 - 0.03)^t = 15(0.97)^t$

2. Suppose a quantity $P$ is an exponential function in time in hours.

\[ P(t) = P_0 a^t \]

and $P = 320$ when $t = 5$ and $P = 500$ when $t = 3$.

Find $a_0$ and $P_0$

\[
\begin{align*}
320 &= P(5) = P_0 a^5 \quad \text{...(i)} \\
500 &= P(3) = P_0 a^3 \quad \text{...(ii)}
\end{align*}
\]

We now have 2 equations and 2 unknowns.

Divide (i) by (ii)

\[
\frac{320}{500} = \frac{P_0 a^5}{P_0 a^3} = a^2 \Rightarrow a^2 = \frac{16}{25} \Rightarrow a = \frac{4}{5}
\]
To find $P_0$, from (5):

$$500 = P_0 \left( \frac{4}{5} \right)^3 \Rightarrow 500 = P_0 \left( \frac{64}{125} \right) \Rightarrow P_0 = \frac{500 \cdot 125}{64} = 976.56$$

The formula is:

$$P(t) = 976.56 \left( \frac{4}{5} \right)^t$$

problem 22.

We have $P(t) = 976.56 \left( 1 - 0.2 \right)^t$ so the quantity $P$ is decreasing at an hourly percentage rate of 20%.
### Exponential vs Linear

<table>
<thead>
<tr>
<th>Exponential</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P = P_0 e^{at} )</td>
<td>( y = mt + b )</td>
</tr>
</tbody>
</table>

- **Exponential:** Constant percentage change.
- **Linear:** Constant rate of change.
- The population is increasing by 5% a year.
- The population is increasing by 500 people per year.

#### Family of Exponential Functions

- \( y = 2^t \)
- \( y = e^t \) \((e = 2.718)\)
- \( y = 2^t \)
- \( y = (1.5)^t \)

\( P(t) = a^t \) (growth)

As the base increases, the rate of growth increases.

#### Exponential Decay

- \( y = (0.95)^t \)
- \( y = (0.9)^t \)
- \( y = (0.8)^t \) decreases faster on a decrease.

\( P = a^t \) (Decay)
Laws of exponents

1. \( x^m \cdot x^n = x^{m+n} \)

2. \( \frac{x^m}{x^n} = x^{m-n} \)

3. \( (x^m)^n = x^{mn} \)

4. \( \left( \frac{x}{y} \right)^n = \frac{x^n}{y^n} \)

5. \( x^{-n} = \frac{1}{x^n} \)

6. \( (\frac{x}{y})^{-n} = \left( \frac{y}{x} \right)^n \)

7. \( x^0 = 1 \)

8. \( x^1 = x \)

Example: They do

\[ W = 39,295 \text{ Mga Naka} \text{ and } 120,908 \text{ in } 2003 \text{ and } 2008. \]

The constant \( e = 2.718 \) yields a special exponential function

\[ y = e^t. \]

If \( y = e^{2t} \), using the laws of exponents we can write

this function in the form \( y = a^t \). How?

\[ y = e^{2t} \Rightarrow y = (e^2)^t \approx (7.389)^t. \]