1.5 Exponential Functions

$$f(x) = 2^{x}$$ ← the power is a variable x.

↑ base

In general, exponential functions take the form $f(x) = ka^{x}$.

$f(0) = ka^{0} = k$. i.e $a^{0} = 1$ for any constant a.

Exponential functions are ideal for describing population growth

e.g

YEAR	Population (Millions)	Change in population	Ratio of pop
2003	12.853		
2004	13.290	0.437	1.034
2005	13.747	0.457	1.034
2006	14.225	0.478	1.034

→ $\frac{13.290}{12.853}$

Remarks

1. The rate of growth of the population is not constant ⇒ the population growth is NOT linear

2. The ratio of population in successive years is constant.

If we let t be the time in years since 2003, then

$P(0) = 12.853$

$\frac{P(1)}{P(0)} = 1.034 \Rightarrow \boxed{P(1) = P(0) \cdot 1.034}$

$\frac{P(2)}{P(1)} = 1.034 \Rightarrow P(2) = P(1) \cdot 1.034$ but $P(1) = P(0) \cdot 1.034$

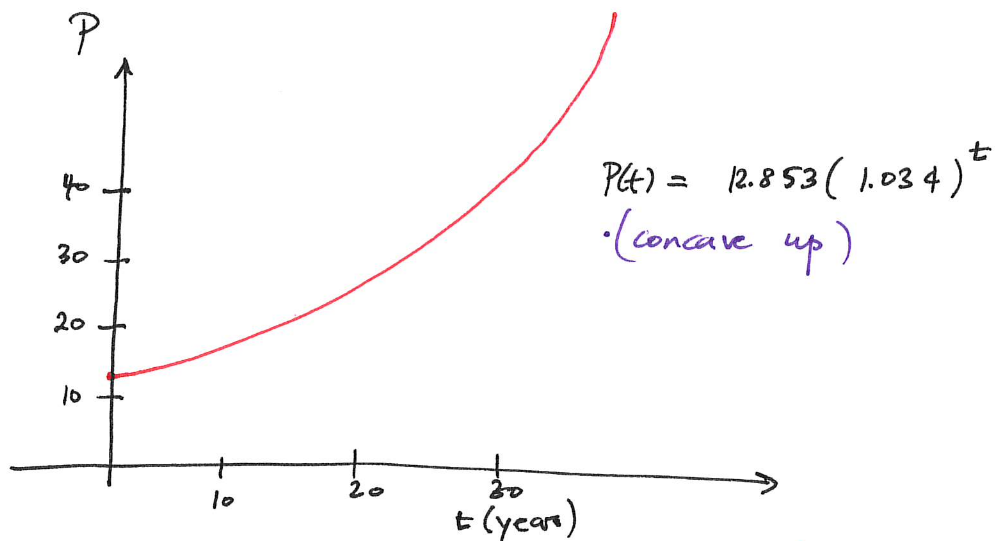
so $P(2) = P(0) \cdot (1.034)^{2}$

$\vdots$

$P(t) = 12.853 \cdot (1.034)^{t}$ million. [The population is growing at 3.4% per YEAR].

$$P(t) = \underbrace{12.853}_{\text{Initial population}} \cdot \underbrace{(1.034)}_{\text{growth factor}}^{t}$$

growth factor (amt by which pop grows each year)



- The population is growing faster as time passes.
- In general, exponential functions increase slowly at first then rise quickly

Exponential Decay

Suppose we start with an initial dose of 250mg of drug in the blood.
Suppose the concentration of drug decreases by 40% per hour.

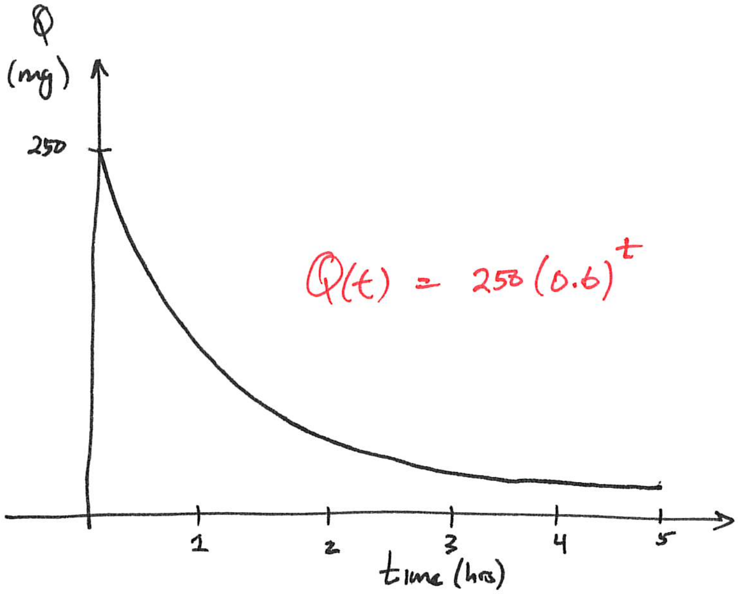
Let $Q(t)$ be the concentration of drug at time t.

$$Q(0) = 250\text{mg}$$
$$Q(1) = 250 \cdot (1 - 0.4) = 250 \cdot 0.6$$
$$Q(2) = Q(1) \cdot 0.6 = 250 \cdot (0.6)^2$$
$$Q(3) = Q(2) \cdot 0.6 = 250 \cdot (0.6)^3$$

so after t hours, the concentration of drug is

$$Q(t) = 250 \cdot (0.6)^t$$

This is called exponential decay.



The graph is decreasing and concave up.

In general exponential functions take the form

$$P = P_0 a^t$$

where

1. P_0 is the initial amount
2. a is the factor by which P changes when t increases by 1
   - $\Rightarrow$ $a > 1$ means we have exponential growth
   - $\Rightarrow$ $0 < a < 1$ means we have exponential decay.
3. $a = 1 + r$, where r is a percentage rate of change

In our examples.

1. $P(t) = P_0 (1.034)^t \Rightarrow a = 1.034 = 1 + 0.034 \Rightarrow r = 3.4\%$
2. $Q(t) = Q_0 (0.6)^t = a = 1 - 0.4 \Rightarrow r = 40\%.$

Examples

1. The initial amount of drug in a body is 15mg. Find a formula for A, the amount in mg, at a time t minutes later if A is

(a) Decreasing by 0.4mg per min (Constant rate of change)

(b) Decreasing by 3% per minute. (Constant percentage change)

(a) $A(t) = 15 - 0.4t$

(b) $A(t) = 15(1-0.03)^t = 15(0.97)^t$

If Quantities were <u>increasing</u> then

(a) $A(t) = 15 + 0.4t$

(b) $A(t) = 15(1+0.03)^t = 15(1.03)^t$

2. Suppose a quantity P is an exponential function in time in hours.

i.e $P(t) = P_0 a^t$

and $P = 320$ when $t = 5$ and $P = 500$ when $t = 3$.

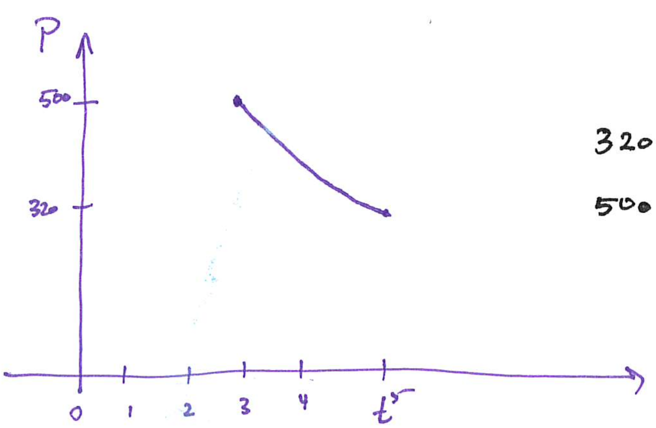
Find a_0 and P_0

$$\boxed{P(5) = 320 \text{ and } P(3) = 500.}$$

$$\Downarrow$$

$$320 = P(5) = P_0 a^5 \quad \ldots (i)$$

$$500 = P(3) = P_0 a^3 \quad \ldots (ii)$$



We now have 2 equations and 2 unknowns.

$$320 = P_0 a^5 \quad \ldots (i)$$

$$500 = P_0 a^3 \quad \ldots (ii)$$

$\Rightarrow$ Divide (i) by (ii)

$$\frac{320}{500} = \frac{P_0 a^5}{P_0 a^3}$$

$$\frac{32}{50} = a^2 \Rightarrow a^2 = \frac{16}{25} \Rightarrow a = \frac{4}{5}$$

To find P_0, from (ii)

$$500 = P_0\left(\frac{4}{5}\right)^3 \Rightarrow 500 = P_0\left(\frac{64}{125}\right) \Rightarrow P_0 = 500 \cdot \frac{125}{64} = \underline{976.56}$$

the formula is

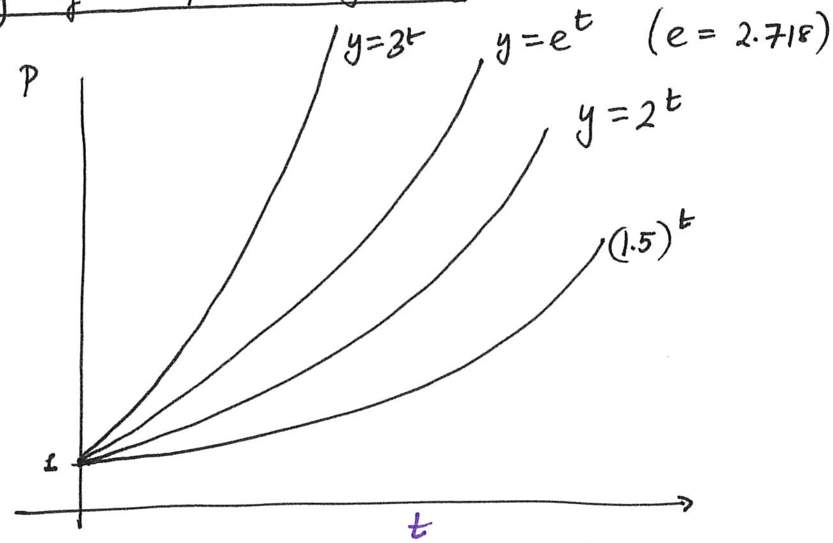
$$\boxed{P(t) = 976.56\left(\frac{4}{5}\right)^t}$$

problem 22.

We have $P(t) = 976.56(1-0.2)^t$ so the quantity p is decreasing at an hour percentage rate of 20%.

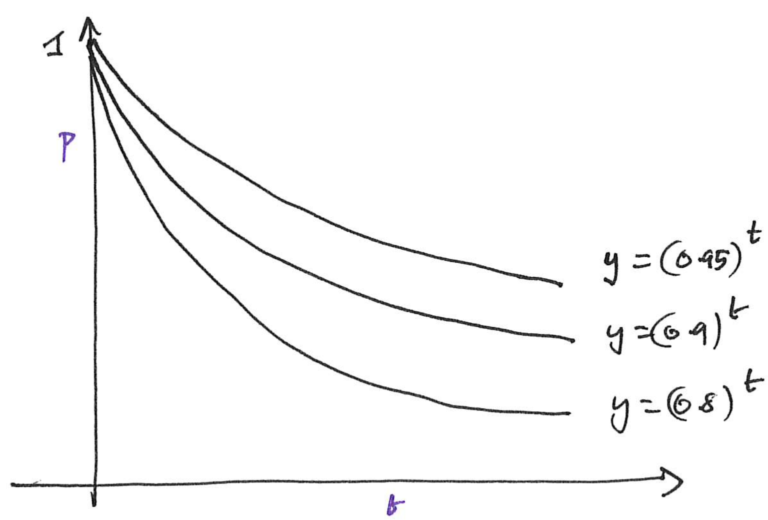
Exponential	vs linear
$P = P_0 a^t$	$y = mt + b$
constant percentage change	constant rate of change.
* The population is increasing by 5% a year	The population is increasing by 500 people per year.

Family of Exponential functions



as the base increases the rate of growth increases.

$P(t) = a^t$. (growth)



decreases faster as a decreases

$P = a^t$ (Decay)

Laws of exponents

1. $x^m \cdot x^n = x^{m+n}$

2. $\dfrac{x^m}{x^n} = x^{m-n}$

3. $(x^m)^n = x^{mn}$

4. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

5. $x^{-n} = \dfrac{1}{x^n}$

6. $\left(\dfrac{x}{y}\right)^{-n} = \left(\dfrac{y}{x}\right)^n$

7. $x^0 = 1$

8. $x^1 = x.$

Example: They do

#2

$W =$ 39,295 Mga Watts (2003) and 120,908 (2008).

#26, #25

The constant $e = 2.718$ yeilds a special exponential function

$$y = e^t.$$

If $y = e^{2t}$, using the laws of exponents we can write this function in the form $y = a^t$. How?

$$y = e^{2t} \Rightarrow y = (e^2)^t \approx (7.389)^t.$$