1.6

THE NATURAL LOG FUNCTION.

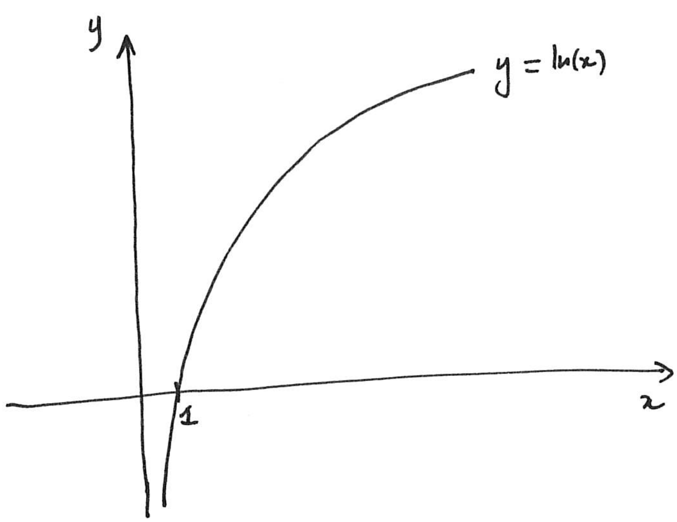
Suppose $P(t) = 2.020(1.036)^t$ is the population of a town in millions. When does the population reach 4 million? * Note that t is time since 2000.

$$4 = 2.020(1.036)^t$$

$$\boxed{\frac{4}{2.02} = (1.036)^t}$$

To solve this equation we recall the Natural log function.

i.e $\ln(x) = c \Rightarrow e^c = x$.



e.g $\ln(e^3) = 3$

$\ln(x)$ is defined for $x > 0$

LAWS OF LOGS

1. $\ln(AB) = \ln A + \ln(B)$
2. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
3. $\ln(A^p) = p\ln A$
4. $\ln(e^x) = x$
5. $e^{\ln(x)} = x$
6. $\ln(1) = 0$ because $e^0 = 1$.

Example

$P = 25(0.88)^t$ represents a 12% annual decay rate.
(t measured in years)

We can write this in the form $P = P_0 e^{kt}$ as

$25(0.88)^t = 25(e^k)^t \Rightarrow e^k = 0.88 \Rightarrow k = \ln(0.88) = -0.127.$

So $P(t) = 25e^{-0.127t} \Rightarrow$ 12.7% ~~dec~~ continuous decay rate.

We can also work in the other direction i.e.

given $P(t) = 15e^{0.25t} \Rightarrow$ 25% continuous growth rate.

We can write this in the form $P(t) = P_0 a^t$

Indeed $15e^{0.25t} = 15a^t \Rightarrow a = e^{0.25} = 1.28$

So $P(t) = 15(1.28)^t \Rightarrow$ 28% annual growth rate.

Inclass example (Try)

Solve for t using ln

1) (i) $2 = 1.02^t$ (ii) $5e^{3t} = 8e^{2t}$

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Back to our problem

$$4 = 2.02\,(1.036)^t$$

$$\frac{4}{2.02} = (1.036)^t$$

$$\ln\left(\frac{4}{2.02}\right) = \ln(1.036)^t$$

$$\ln\left(\frac{4}{2.02}\right) = t\,\ln(1.036)$$

so $$t = \frac{\ln\left(\frac{4}{2.02}\right)}{\ln(1.036)} = 19.317 \text{ years.}$$

In 2019, the population reaches 4 million.

Exponential functions with base e

In general $$P(t) = P_0\, a^t.$$

for any base a, we can write a^t as e^k:

Indeed $a = e^k \Rightarrow k = \ln(a)$.

so that $$P(t) = P_0\, e^{kt}$$

$\boxed{\text{KEY}}$ If $0 < a < 1$ (exponential decay) $k < 0$

$a > 1$ (exponential growth) $k > 0$

k is called the continuous growth rate.