1.7

Exponential Growth and Decay

$$P(t) = P_0 e^{kt}$$ (exponential growth / decay function)

describes naturaly growing / decaying quantities.

k – continuous growth / decay rate.

Financial Applications

1. Compounding interest:

- Suppose \$100 is deposited into a bank paying interest at 8% per year.
- The amount of interest depends on how often the interest is compound.

e.g (a) After 1 year, if it is compounded annually

$$P(t) = 100(1+0.08)^t = \boxed{\$108} \text{ @ } t=1.$$

(b) Semi-annually. – interest is paid twice. (4% installments)

$$P(t) = P_0\left(1+\frac{0.08}{2}\right)^{2t}$$

$$P(1) = 100(1.04)^2 = \$108.16$$

(c) Quarterly

$$P(t) = P_0\left(1+\frac{0.08}{4}\right)^{4t}$$

$$P(1) = P_0(1.02)^4 = \$108.24$$

(d) Daily

$$P(t) = P_0\left(1+\frac{0.08}{365}\right)^{365t}$$

$$P(1) = 100(1.0002)^{365} = \$108.32.$$

The more frequently the compounding, the better!

If the interest is compounded n times, then $P(t) = P_0\left(1+\frac{r}{n}\right)^{nt}$, r is an annual interest rate.

If the interest is compounded continuously, then

$$P(t) = P_0 e^{rt}, \text{ where } e = 2.71828.$$

$$P_0\left(1+\frac{r}{n}\right)^{nt} \text{ as } n \to \infty$$

In this case, if the 8% is compounded continuously, $P(1) = 100e^{0.08 \cdot 1}$
$= \underline{\$108.33}$

Example

If \$10,000 is deposited in an account paying 5% per year, compounding continuously.

$$\boxed{P(t) = 10{,}000\, e^{0.05t}}$$

(i) How long does it take for the balance to reach \$15,000?

$$15{,}000 = 10{,}000\, e^{0.05t}$$

$$1.5 = e^{0.05t}$$

$$\ln(1.5) = 0.05t \quad \Rightarrow \quad t = \frac{\ln(1.5)}{0.05} = 8.1093$$

It takes 8.1 years for the account to reach \$15,000.

(ii) How long does it take for the balance to double (doubling time)

$$20{,}000 = 10{,}000\, e^{0.05t}$$

$$2 = e^{0.05t} \quad \Rightarrow \quad t = \frac{\ln(2)}{0.05} = \underline{13.86 \text{ years}}$$

(iii) If the compounding is annual,

$$P(t) = 10{,}000\,(1+0.05)^t$$
$$= 10{,}000\,(1.05)^t$$

$$15{,}000 = 10{,}000\,(1.05)^t$$
$$1.5 = (1.05)^t \quad \Rightarrow$$
$$\ln(1.5) = t\ln(1.05) \quad \Rightarrow \quad t = \frac{\ln(1.5)}{\ln(1.05)} = 8.31 \text{ years.}$$

Doubling time

$$20{,}000 = 10{,}000(1.05)^t$$

$$2 = (1.05)^t \Rightarrow t = \frac{\ln(2)}{\ln(1.05)} = \underline{14.206 \text{ years.}}$$

$D \simeq \frac{70}{r} \simeq 14$ annual compounding

In general

(i) Doubling time of an exponentialy increasing quantity is the time required for the quantity to double

(ii) Half-life of an exponentaly decreasing function is the time to reduce by a factor of half.

PRESENT AND FUTURE VALUES.

- Future value, B of a payment, P, is the amount P would have grown if deposited today into an interest bearing account.
- Present Value, P of a future payment, B, is the amount that would have to be deposited in a bank to produce exactly B in the future.

Compounding annually

$$B = P(1+r)^t$$ (B is the future value of P dollars)

(*) P is the present value of B,

$$P = \frac{B}{(1+r)^t}$$

** Compounding continuously

$$B = Pe^{rt} \Rightarrow P = \frac{B}{e^{rt}} = Be^{-rt}$$

r is also called the discount rate.

Example

You win the lottery and are offered the choice between \$1,09000 in 4 yearly installments of \$250,000 or a lump sum of \$920,000.

Assuming a 6% interest rate compounded continuously, which is the best choice?

Comparing future values

After 3 years \$92,0,000 is worth

$$\$920{,}000\, e^{0.06(3)} = \boxed{\$1{,}101{,}440}$$

the installments are worth

$$\$250{,}000\, e^{0.06(3)} + \$250{,}000\, e^{0.06(2)} + \$250{,}000\, e^{0.06(1)} + \$250{,}000$$

$$\boxed{= \$1{,}096{,}637}$$

It is better to take the lump-sum!