Exponential Growth and Decay

\[ P(t) = P_0 e^{kt} \]  
(exponential growth/decay function)

describes materially growing/decaying quantities.

\( k \) - continuous growth/decay rate.

Financial Applications

1. Compounding interest.

Suppose $100 is deposited into a bank paying interest at 8\% per year.

- The amount of interest depends on how often the interest is compounded.

  e.g. (a) After 1 year, if it is compounded annually

  \[ P(t) = 100 \left( 1 + 0.08 \right)^t = \$108 \text{ at } t=1. \]

  (b) Semi-annually - interest is paid twice. (4% installment)

  \[ P(t) = P_0 \left( 1 + \frac{0.08}{2} \right)^{2t} \]

  \[ P(1) = 100 \left( 1.04 \right)^2 = \$108.16 \]

  (c) Quarterly

  \[ P(t) = P_0 \left( 1 + \frac{0.08}{4} \right)^{4t} \]

  \[ P(1) = P_0 \left( 1.02 \right)^4 = \$108.24 \]

  (d) Daily

  \[ P(t) = P_0 \left( 1 + \frac{0.08}{365} \right)^{365t} \]

  \[ P(365) = 100 \left( 1.0002 \right)^{365} = \$108.32. \]

The more frequently the compounding, the better!

If the interest is compounded \( n \) times, then

\[ P(t) = P_0 \left( 1 + \frac{r}{n} \right)^{nt} \]

\( r \) is an annual interest rate.
If the interest is compounded continuously, then

\[ P(t) = P_0 e^{rt}, \text{ where } e = 2.71828. \]

In this case, if the 80% is compounded continuously, \( P(1) = 100e^{0.08 \cdot 1} = 108.33 \)

**Example**

If $10,000 is deposited in an account paying 5% per year, compounding continuously,

\[ P(t) = 10,000 e^{0.05t} \]

(i) How long does it take for the balance to reach $16,000?

\[ 16,000 = 10,000 e^{0.05t} \]

\[ 1.5 = e^{0.05t} \]

\[ \ln(1.5) = 0.05t \quad \Rightarrow \quad t = \frac{\ln(1.5)}{0.05} = 8.693 \]

It takes 8.1 years for the account to reach $16,000.

(ii) How long does it take for the balance to double (doubling time)

\[ 20,000 = 10,000 e^{0.05t} \]

\[ 2 = e^{0.05t} \quad \Rightarrow \quad t = \frac{\ln(2)}{0.05} = 13.86 \text{ years} \]

(iii) If the compounding is annual,

\[ P(t) = 10,000 (1 + 0.05)^t \]

\[ 15,000 = 10,000 (1.05)^t \]

\[ 1.5 = (1.05)^t \quad \Rightarrow \quad t = \frac{\ln(1.5)}{\ln(1.05)} = 8.31 \text{ years} \]
Doubling time

\[ 20,000 = 10,000 \cdot (1.05)^t \]

\[ 2 = (1.05)^t \Rightarrow t = \frac{\ln(2)}{\ln(1.05)} = 14.286 \text{ years.} \]

In general

(i) Doubling time of an exponentially increasing quantity is the time required
for the quantity to double.

(ii) Half-life of an exponentially decreasing function is the time to reduce
by a factor of half.

PRESENT AND FUTURE VALUES.

- Future value, \( B \) of a payment, \( P \), is the amount \( P \) would have grown if
deposited today into an interest bearing account.

- Present Value, \( P \) of a future payment, \( B \), is the amount that would have to
be deposited in a bank to produce exactly \( B \) in the future.

Compounding annually

\[ B = P(1+r)^t \quad (B \text{ is the future value of } P \text{ dollars}) \]

\( P \) is the present value of \( B \),

\[ P = \frac{B}{(1+r)^t} \]

Compounding continuously

\[ B = Pe^{rt} \Rightarrow P = \frac{B}{e^{rt}} = Be^{-rt} \]

\( r \) is also called the discount rate.
Example
You win the lottery and are offered the choice between $1,000,000 in 4 yearly installments of $250,000 or a lump sum of $920,000.

Assuming a 6% interest rate compounded continuously, which is the best choice?

Comparing future values

After 3 years, $920,000 is worth

\[ 920,000 e^{0.06 \times 3} = 920,000 e^{0.18} = 1,101,440 \]

The installments are worth

\[ 250,000 e^{0.06 \times 3} + 250,000 e^{0.06 \times 2} + 250,000 e^{0.06 \times 1} + 250,000 = 1,096,637 \]

It is better to take the lump sum!