2.3 Interpretation of the derivative

Derivative — slope & rate of change

NOTATION

(i) $f'(x)$ - derivative of $y = f(x)$

→ measures the rate of change of y w.r.t. respect to x.

(ii) $f'(x) \cong \frac{\Delta y}{\Delta x}$ "$\frac{\text{change in } y}{\text{change in } x}$" $\Rightarrow$ $f'(x) = \frac{dy}{dx}$

$\frac{dy}{dx}$ - means the derivative of y with respect to x.
- the rate of change of y with respect to x

UNITS

If $s = f(t)$ is the position in meters at time, t

$f'(t) = \frac{ds}{dt}$ is the rate of change of distance w.r.t. respect to time.

$f'(2) = 10$ m/s $\Rightarrow$ @ $t=2$, the object is moving at 10m/s.

* In general,

the units of the derivative of a function are the units of the dependent variable divided by the units of the independent variable.

eg $y = f(t)$

Units of $f'(t)$ are $\frac{\text{units of } y}{\text{units of } t}$.

If the derivative of a function is not changing rapidly near a point, then the derivative is approximately equal to the change in the function if the independent variable increases by 1.

$$f'(x) = \frac{dy}{dx} \cong \frac{\Delta y}{\Delta x}$$

If $\Delta x = 1$, then $\Delta y \cong f'(x)$ at the point x.

$$\boxed{f(x+1) - f(x) \cong f'(x).}$$

Example #1

The cost ~~to~~ of building a property ($) depends on the sur'area (m^2)

$$C = f(A)$$

What are the units an interpretation of $f'(A)$?

$$f'(A) = \frac{dC}{dA}.$$

dC is the extra cost of adding dA m^2.

If you are building a house of size A square metres, then

$f'(A) \cong$ cost per square metre to build a house that is 1 foot larger.

e.g. $f'(1000) = 5 \Rightarrow$ If you are building a $1000\,m^2$ house it will cost $\$5/m^2$ to build one that is $1001\,m^2$.

Example #2

Cost of mining T tons of ore is given by

$$C = f(T) \text{ dollars.}$$

(T - tons, C - dollars)

$f'(2000) = 150.$

$\frac{dC}{dT}$ @ $T = 2000$ = \$150. dC is the extra cost of producing dT tons.

When 2000 tons have been mined, it costs \$150 to get the 2001st ton.

Example #3

$q = f(p)$ describes the impact of price on sales (aka demand equation).

Interpret

(i) $f(150) = 2000$ and (ii) $f'(150) = -25$.

$$\boxed{\underset{\text{quantity sold (items)}}{q} = f(\underset{\text{price per unit in dollars}}{p})}$$

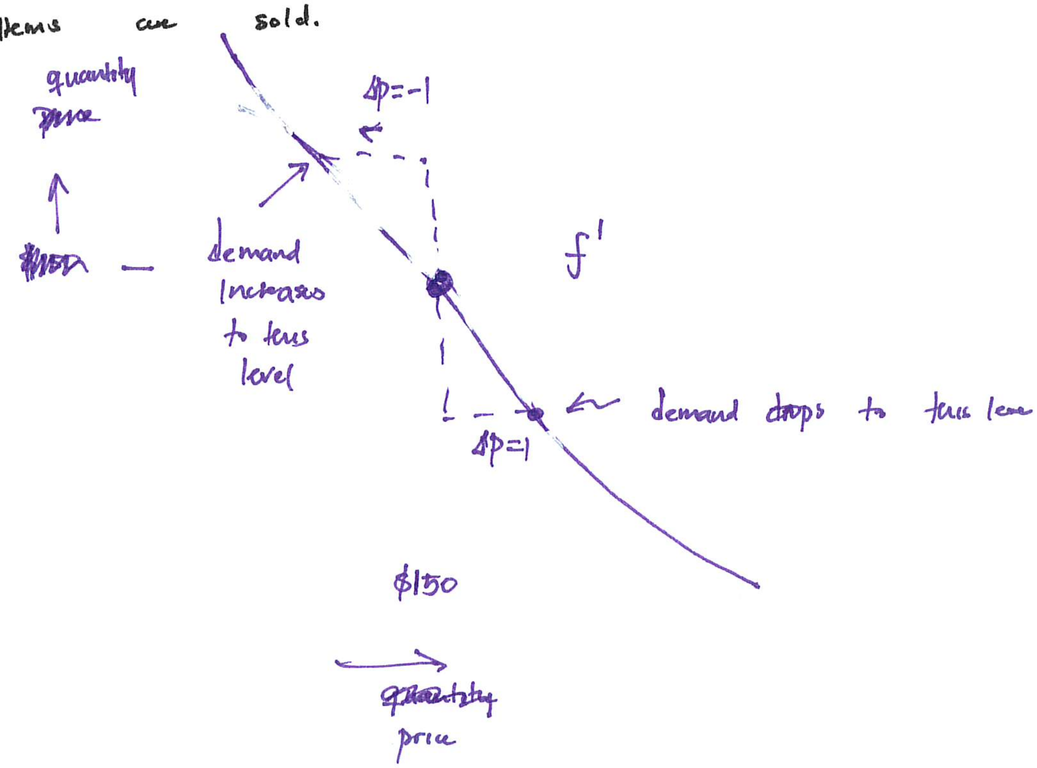
(i) When the price is \$150, 2000 items are sold.

(ii) $f'(150) = -25$.

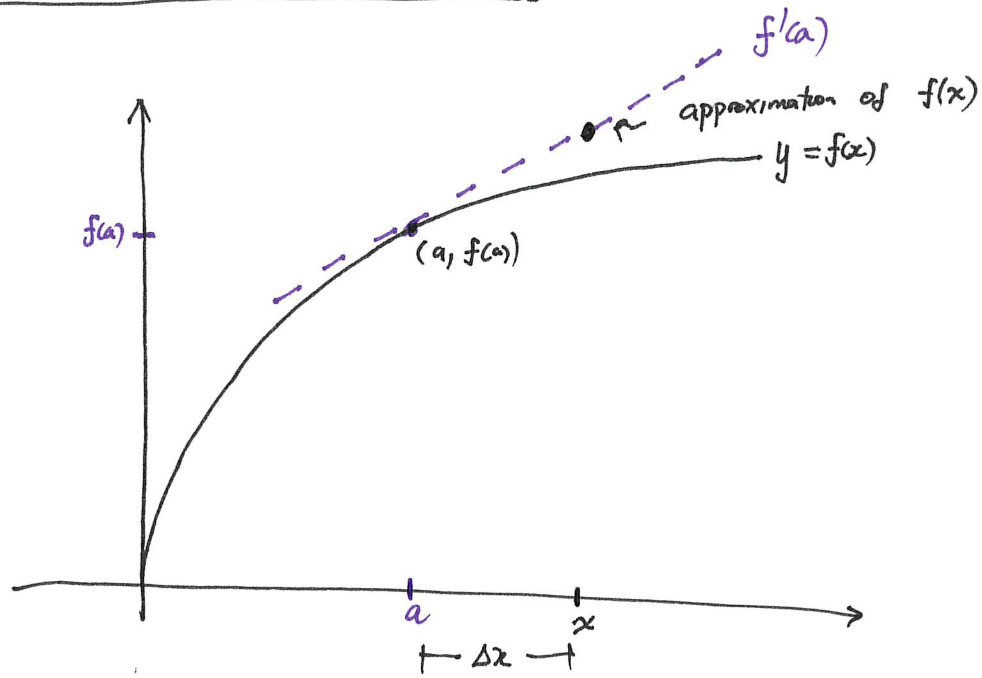
$$\boxed{f'(p) = \frac{dq}{dp} = \frac{\text{change in quantity sold}}{\text{change in price.}}}$$

* $f'(150) = -25 \Rightarrow$ If the price goes up from \$150 (by \$1) to \$151, 25 fewer items are sold.

** If the price goes down from \$150 to \$149, 25 more items are sold.



Using the derivative to approximate function values



GOAL

Given $f(a)$ and $f'(a)$, we want to estimate f at points nearby, eg x.

Strategy

We assume that the function is linear near $x = a$.

linear with slope $m = f'(a)$

so
$$f'(a) \cong \frac{f(x) - f(a)}{x - a} \Rightarrow f(x) \cong f(a) + f'(a)(x-a)$$

$$\boxed{f(x) \cong f(a) + f'(a)\,\Delta x}$$

Remarks

1. $f(x) \cong \underbrace{f(a)}_{\text{initial value}} + \underbrace{f'(a)\,\Delta x}_{\text{move in direction of slope}}$ distance $= $ "Δx".

2. This is also called the tangent line approximation or local linearity

3. The approximation is realiable when x is close to a.

Relative rate of change

$$\boxed{\begin{array}{l}\text{Given } y = f(t), \\ \text{relative rate of change of } y \text{ @ } t=a = \dfrac{\frac{dy}{dt}}{y} = \dfrac{f'(a)}{f(a)}\end{array}}$$

the relative rate of change is a continuous percentage change

Example #1

The area of Brazil's rain forest is $R = f(t)$, in million acres, is a function of the number of years t, since 2010.

Given $f(9) = 740$ and $f'(9) = -2.7$

Relative rate of change @ $t=9$ $= \dfrac{-2.7}{740} = -0.00364$

a a continuous rate of

In 2019, the rainforest will decrease ~~by~~ 0.364%. per year.

Example #2

The number of barrels of oil produced is given by

$$B = 21.88\,e^{0.059t} \text{ million barrels}$$

when t is time measured in months since Sept 2012.

Estimate the relative change in oil production in December 2012 using $\delta t = 1$.

$$\frac{\frac{dB}{dt}}{B} = \frac{B'(3)}{B(3)} \cong \frac{1}{B(3)} \cdot \frac{B(4) - B(3)}{1}$$

$$= \frac{1}{21.88\,e^{0.059(3)}} \left(\frac{21.88\,e^{0.059 \cdot 4} - 21.88\,e^{0.059 \cdot 3}}{1} \right)$$

Example

A cornell university study on maize production in Kenya found that the average value $y = f(x)$ in Kenyan shillings of yearly maize production from an average plot of land is a function of the quantity, x, of fertilizer used in kg.

$$\underset{\text{maize output (shillings)}}{y} = f(\overset{\text{fertilizer (kg).}}{x})$$

(a) (i) Interpret $f(5) = 11{,}500$ and $f'(5) = 350$

(b) Estimate $f(6)$. $f(10)$ (Which estimate is more reliable)

(a) (i) $f(5) = 11{,}500$

$y =$ ~~\$~~ $11{,}500$ when $x = 5$.

If 5kg of fertilizer are applied maize worth 11,500 Kenyan shillings is produced.

~~(b)~~ (ii) $\dfrac{dy}{dx} = 350$ when $x = 5$.

If the amount of fertilizer used is 5kg and increases by 1kg, then the maize production increases by 350 Kenyan shillings.

(b) $f(6) \cong f(5) + f'(5)(6-5)$

$11{,}500 + 350 \cdot 1$

$11{,}850$ Kenyan shillings.

$f(10) = f(5) + f'(5)(10-5)$

$11{,}500 + 350 \cdot 5$

$11.500 + 1750 = \underline{11{,}850}$

$f(6)$ is more ~~less~~ reliable than $f(10)$.