

Numerical Integration

Spring 2019

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A jog down Calc I/II lane

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- In some practical cases, we do not have an analytical representation of f but we still want to approximate the integral
- Numerical integration techniques are necessary to approximate the integral

Approximating $\int_a^b f(x) dx$ (basic idea)

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$$\Delta x = \frac{b - a}{n}$$

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- More sophisticated methods use adaptive widths of subinterval depending on the behavior of the function
- As the number of sub-intervals increases, we obtain a more accurate approximation of the area under the curve

▶ [Desmos demo](#)

Approximating $\int_a^b f(x) dx$ (Implementation)

Rectangles

- 1 Divide $[a, b]$ so that

$$a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b, \quad k = 1, 2, \dots, n$$

with $x_k = a + (k - 1)\Delta x$

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- 2 On each sub-interval $[x_k, x_{k+1}]$ select a *sample point*, $x_k^* \in [x_k, x_{k+1}]$

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- 2 On each sub-interval $[x_k, x_{k+1}]$ select a *sample point*, $x_k^* \in [x_k, x_{k+1}]$
- 3 Define the height of each **sub-rectangle** as $f(x_k^*)$ so that the area of each sub-rectangle is

$$f(x_k^*)\Delta x$$

- 4 Summing up for the n sub-intervals

$$\int_a^b f(x) dx \approx \sum_{k=1}^n f(x_k^*)\Delta x$$

Approximating $\int_a^b f(x) dx$ (Implementation)

Trapezoids

- 1 Each trapezoid has a base of $[x_k, x_{k+1}]$ with parallel sides of length $f(x_k)$ and $f(x_{k+1})$

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- 2 The area of the k -th Trapezoid is

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Trapezoids

$$\begin{aligned}\int_a^b f(x) dx &\approx \frac{\Delta x}{2} \sum_{k=1}^n (f(x_k) + f(x_{k+1})) \\ &= \frac{\Delta x}{2} ([f(x_1) + f(x_2)] + [f(x_2) + f(x_3)] + \cdots + [f(x_{n-1}) + f(x_n)] + [f(x_n) + f(x_{n+1})]) \\ &= \frac{\Delta x}{2} (f(x_1) + 2f(x_2) + \cdots + 2f(x_n) + f(x_{n+1}))\end{aligned}$$

$$\int_a^b f(x) dx = \frac{\Delta x}{2} \left(f(x_1) + 2 \sum_{k=2}^n f(x_k) + f(x_{n+1}) \right)$$

Error Analysis (MA 428)

Theorem

Assuming $\max_{a \leq x \leq b} |f''(x)| \leq M$. Then the midpoint method has error

$$\frac{M(b-a)(\Delta x)^2}{24}$$

and the Trapezoidal method has error

$$\frac{M(b-a)(\Delta x)^2}{12}$$

Simpson's Method for approximating $\int_a^b f(x) dx$

Approximate the area under the function using second order curves

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- 1 Divide $[a, b]$ into n sub-intervals of width $\Delta x = \frac{b-a}{n}$, where n is **even**.

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Approximate the area under the function using second order curves

- 1 Divide $[a, b]$ into n sub-intervals of width $\Delta x = \frac{b-a}{n}$, where n is **even**.
- 2 On **each pair** of sub-intervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ ($k = 2, \dots, n$) approximate the area under the curve with a **quadratic** function passing through the points:

$$\boxed{(x_{k-1}, f(x_{k-1})), (x_k, f(x_k)) \text{ and } (x_{k+1}, f(x_{k+1}))}$$

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$$(x_{k-1}, f(x_{k-1})), (x_k, f(x_k)) \text{ and } (x_{k+1}, f(x_{k+1}))$$

- 3 The area under each parabola on $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ is

$$\frac{\Delta x}{3} (f(x_{k-1}) + 4f(x_k) + f(x_{k+1}))$$

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$$\frac{\Delta x}{3} (f(x_{k-1}) + 4f(x_k) + f(x_{k+1}))$$

- 4 Summing up over all sub-intervals

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3} (f_1 + 4f_2 + 2f_3 + 4f_4 + \dots + 2f_{n-1} + 4f_n + f_{n+1})$$

Error Analysis (MA 428)

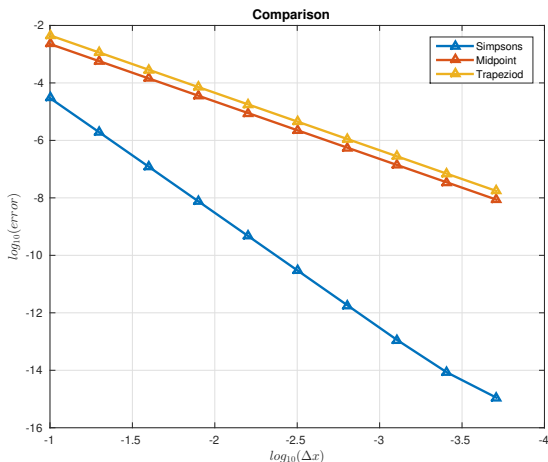
Theorem

Assuming $\max_{a \leq x \leq b} |f^{(4)}(x)| \leq M$. Then the Simpson's method has error

$$\frac{M(b-a)(\Delta x)^4}{180}$$

- Composite Simpson's method has a convergence rate of $\mathcal{O}(\Delta x)^4$ compared to Midpoint and Trapezoidal that are $\mathcal{O}(\Delta x)^2$.

Error comparison



- Bisection and Trapezoidal methods are second order in Δx i.e. $\mathcal{O}((\Delta x)^2)$
- Simpson's method is fourth order in Δx i.e. $\mathcal{O}((\Delta x)^4)$

Generalized formula

Higher order methods of the form

$$\int_a^b f(x) dx = \sum_{i=1}^N f(x_i) w_i$$

- These methods can be extended to 2D and 3D integrals (see MA 428).