Instructions:
The code submitted should be your own creation. You may consult MATLAB's documentation or the notes from class. The submission of codes obtained from online sources is a violation of Loyola's honor code and may result in failure of the course.

Problem 1: Estimating π - Arcus Tangens

John Machin (1680-1752) discovered the following identity

$$\pi = 16\arctan\left(\frac{1}{5}\right) - 4\arctan\left(\frac{1}{239}\right)$$

For centuries, Machin's formula served as a primary method for finding digits of π. In fact, as late as 1973, one million digits of π were evaluated by Guilloud and Bouye (Arndt, J and C. Haenel (2001). *Pi-Unleashed*. Springer) using a version of Machin's formula. Write a function to approximate π using the following approximation.

$$\pi \approx T_n = 16P_n\left(\frac{1}{5}\right) - 4P_n\left(\frac{1}{239}\right)$$

where,

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1}$$

is the *nth*-degree Taylor polynomial for $\arctan x$ centered at $x = 0$.

Testing
Write a script to test your function as follows:

1. Compute the absolute error, $|T_n - \pi|$ for $n = 1, 3, 5, 7, 9, 11$ and include a table of errors and values of n in your summary file.

Problem 2: Newton's method on the complex plane
In class we discussed **Newton's method** for approximating nonlinear equations of the form $f(z) = 0$. Starting with an initial guess z_0, the method generates a sequence $\{z_n\}$ which usually converges to the root. The method is

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}$$

In this problem you will visualize the performance of Newtons method on the **complex plane**. Consider the function $f(z) = z^3 - 1$. It has three roots, one is real and the other two are complex:

$$r_1 = 1, \quad r_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad r_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Using the Newton's iteration

$$z_{k+1} = z_k - \frac{z_k^3 - 1}{3z_k^2}$$

and **complex arithmetic**, the sequence $\{z_k\}$ will converge to one of the 3 roots depending on the initial guess z_0. In this problem you will write a script to visualize which starting point leads to which of the 3 roots. The set of initial values that lead to convergence of the same root is called the *basin of attraction* of that root. Your script is going to produce the *basin of attraction* for this problem.

1. Create a grid of points on the complex plane $\{z_0 = x + iy| -1 \le x, y \le 1\}$ with 1000 points in each direction. Each point on this grid will be an initial guess.

2. Perform the Newton iteration

$$z_{k+1} = z_k - \frac{z_k^3 - 1}{3z_k^2}$$

for all points in your grid at once for $k = 1, \cdots, 50$. Here we pick 50 to ensure that every point has converged to the root.

3. Assuming that Z= z_{50} is the set of values from your Newton loop, you can call

$$>>\texttt{unique(round(imag(Z)))}$$

to confirm that your initial values have converged to 3 different roots. Explain how the output of the following command shows that all the points have converged to 3 roots.

4. Use the following commands to visualize your basin of attraction

$$>>\texttt{image((round(imag(Z))+2)*20);}$$

Save your image in a file **prob2.jpeg** (your image should look like Figure 1 below)
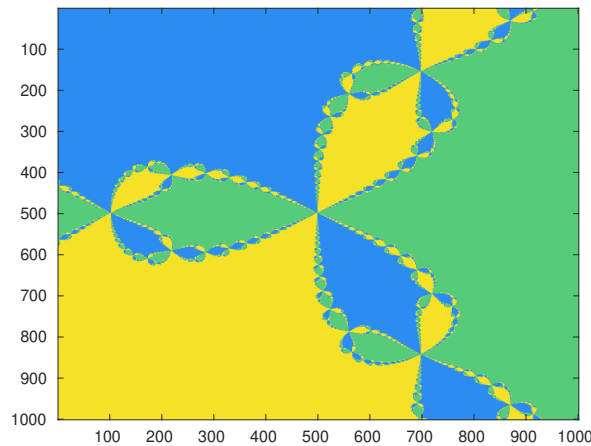

Figure 1: Attraction basin for $f(z) = z^3 - 1$

5. Write commands to determine the number of points on the grid that have converged to each root.

Submit the following:

1. All function and script **m** files.

2. Solutions to questions asked above

3. The image **prob2.jpeg**

Problem 3: Population dynamics – Euler's take

Suppose $x(t)$ is the population of a species (in thousands) and consider the population model

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}$$

The first term on the right-hand side is known as the logistic growth term. This term comes from the assumption that for small population levels, the population will grow at a rate proportional to the current level, while, for large populations, limited resources will cause the growth rate to decrease and eventually become negative. The parameters r and k are called the natural growth rate if the population and the environmental carrying capacity, respectively. The second term on the right-hand side represents harvesting/predation of the species by some other species (e.g. fish being caught by fisherman).

1. For $r = 4$ and $k = 20$, write a script that uses Euler's method to determine the eventual population level reached from an initial population of 2.44 using a time step of $h = 0.1$ weeks.

   Implementation note
   In class we have implemented Euler's method to run on a specific time in a specific interval $[a, b]$. In this case you will need to run Euler's method until the population stops changing, one way to achieve this is to run Euler's method until the difference between the solution at successive time steps is less than 1.0e-06.

2. Repeat part 1, but with the initial population changed to 2.40.

3. Provide a plots (proj_p3a.jph, proj_3b.jpg) of the population in parts (a) and (b) and comment on your results in your summary file.

Submission of exercises

Place all your files (m-files, summary.txt) in a folder named lastname_project and zip the folder to create a file lastname_project.zip. Email your zip file lastname_project.zip to pchidyagwai@loyola.edu with subject MA302_project.