Homework #2

Solutions

1. $y' + y = 5\sin(2t)$

$\mu(t) = e^{\int 1 dt} = e^t$, then $(y'+y)e^t = 5e^t\sin(2t)$

The left hand side always reduces to $\frac{d}{dt}(\mu y) = \frac{d}{dt}(e^t y)$.

$$\frac{d}{dt}(e^t y) = 5e^t \sin(2t)$$

Integrating on both sides yeilds

$$e^t y = \int 5e^t \sin(2t)\,dt$$

Evaluating integral on rhs by parts

$$\int e^t \sin(2t)\,dt = e^t\sin(2t) - 2\int e^t\cos(2t)\,dt$$

$$\boxed{\begin{aligned} u &= \sin(2t) \\ \frac{du}{dt} &= 2\cos(2t) \\ dv &= e^t \\ v &= e^t \end{aligned}}$$

and again!

$$\int e^t \cos(2t)\,dt = e^t\cos(2t) + 2\int \sin(2t)\,dt$$

$u = \cos(2t)$

$\frac{du}{dt} = -2\cancel{\cos}(2t)$ "sin"

$dv = e^t$

$v = e^t$

so

$$\int e^t\sin(2t)\,dt = e^t\sin(2t) - 2\left[e^t\cos(2t) + 2\int \sin(2t)\,dt\right] + C$$

$$5\int e^t \sin(2t)\,dt = e^t\sin(2t) - 2e^t\cos(2t) \Rightarrow \int e^t\sin(2t)\,dt = \frac{1}{5}\left(e^t\sin(2t) - 2e^t\cos(2t)\right)$$

back to the Ode!

$$e^t y = 5\int e^t\sin(2t)\,dt = e^t\sin(2t) - 2e^t\cos(2t) + C$$

so $y(t) = \sin(2t) - 2\cos(2t) + Ce^{-t}$

and $\lim_{t\to\infty}(y(t)) = \sin(2t) - 2\cos(2t)$

2. $ty' + (t+1)y = t$, $y(\ln 2) = 1$, $t > 0$

This is first order and linear but convert into standard form first.

Dividing by t

$$y' + \left(\frac{t+1}{t}\right) y = 1$$

$t > 0$ so $|t| = t$.

$$\mu(t) = e^{\int \frac{t+1}{t} dt} = e^{\int 1 + \frac{1}{t} dt} = e^{t + \ln|t|} = e^t \cdot e^{\ln(t)} = te^t.$$

LHS becomes

$$\frac{d}{dt}\left(te^t \cdot y\right) = te^t$$, integrating on both sides

$$te^t \cdot y = \int te^t dt.$$

$$\int te^t dt = te^t - \int e^t dt = te^t - e^t + c$$

$u = t$ $\quad du = dt$

$dv = e^t$ $\quad v = e^t$

$$te^t \cdot y = e^t(t-1) + c, \qquad y(\ln(2)) = 1 \Rightarrow$$

$$\ln(2) \cdot e^{\ln(2)} \cdot 1 = e^{\ln(2)}\left(\ln(2) - 1\right) + c$$

$$\ln(2) \cdot 2 = 2\left(\ln(2) - 1\right) + c$$

$$2\ln(2) = 2\ln(2) - 2 + c \Rightarrow c = 2. \text{ so}$$

$$te^t y = e^t(t-1) + c^{2}$$

$$y(t) = \frac{e^t(t-1) + 2}{te^t}$$

3. $\frac{dy}{dx} + 2xy = 1$

$\mu(x) = e^{\int 2x\,dx} = e^{x^2}.$

(*) $\frac{d}{dx}\left(e^{x^2} \cdot y\right) = e^{x^2}$, integrating on both sides yeilds

$e^{x^2} \cdot y = \int e^{x^2} dx + C$ so

$y(x) = e^{-x^2}\left[\int e^{x^2} dx + C\right]$

Notice that here $\int e^{x^2} dx$ cannot be expressed by elementary functions so we leave it as is.

(b) To estimate $y(3)$, go back to (*) and integrate from 2 to 3 because we are given $y(2)$ and we need $y(3)$

Indeed,

$$\int_2^3 \frac{d}{dx}\left(e^{x^2} y\right) dx = \int_2^3 e^{x^2} dx$$

$$e^{x^2} y \Big|_{x=2}^{3} = \int_2^3 e^{x^2} dx$$

$\int_2^3 e^{x^2} dx$ can be approximated numerically (eg MATLAB)

$$e^9 \cdot y(3) - e^4 \cdot 1 \cong 1428.09$$

$$y(3) \cong \frac{1428.09 + e^4}{e^9} \cong 0.18\overset{2}{9}.$$

4. $y' + y^2\sin(x) = 0$

$\frac{dy}{dx} = -y^2\sin(x)$, the RHS is separable.

$\frac{dy}{y^2} = -\sin(x)dx$

$\int y^{-2}dy = -\int \sin(x)dx$

$-y^{-1} = \cos(x) + C$

$\frac{1}{y} = C - \cos(x) \Rightarrow y(x) = \frac{1}{C-\cos(x)}.$

5. $\frac{dy}{dx} = xy^3 \cdot \frac{1}{\sqrt{1+x^2}}$

$\int \frac{1}{y^3}\,dy = \int \frac{x}{\sqrt{1+x^2}}$

$-\frac{1}{2}y^{-2} = \sqrt{1+x^2} + C$

$y(0) = 1 \Rightarrow$

$-\frac{1}{2} = \sqrt{1+0^2} + C$

$C = -\frac{3}{2}$

Evaluating RHS integral

$\int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2} + C$

Let $u = 1+x^2$

$\frac{du}{dx} = 2x \Rightarrow \frac{du}{2} = x\,dx$

$\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-\frac{1}{2}}du$

$= u^{\frac{1}{2}} + C$

$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$

$-\frac{1}{2y^2} = \frac{2\sqrt{1+x^2} - 3}{2}$

$\frac{1}{-1\left(2\sqrt{1+x^2} - 3\right)} = y^2$

$y = \pm\sqrt{3 - 2\sqrt{1+x^2}}$, we need $y(0) = 1$ so

$y = \sqrt{3 - 2\sqrt{1+x^2}}.$

$\frac{9}{4} > 1+x^2$

$\therefore$

Solution is defined for $|x| < \frac{\sqrt{5}}{2}$.

$3 - 2\sqrt{1+x^2} > 0 \Rightarrow \frac{3}{2} > \sqrt{1+x^2}$ .... $\sqrt{x^2} < \frac{5}{4}$ so

6 $$\frac{dx}{dt} = 2 - tx^2 - t + 2x^2$$

$$= 2 + 2x^2 - tx^2 - t = 2(1+x^2) - t(x^2+1) = (x^2+1)(2-t)$$

The rhs is separable, therefore

$$\int \frac{dx}{x^2+1} = \int (2-t)dt$$

$$\tan^{-1}(x) = 2t - \frac{t^2}{2} + c$$

$$x = \tan\left(2t - \frac{t^2}{2} + c\right)$$

7. $$\frac{dP}{dt} = rP\left(1 - \frac{P}{N}\right)$$

$$\int \frac{dP}{P(1-\frac{P}{N})} = \int r\,dt$$

The integral on the LHS we use partial fractions

$$\int \frac{1}{P(1-\frac{P}{N})} dP$$

Let $$\frac{1}{P(1-\frac{P}{N})} = \frac{A}{P} + \frac{B}{1-\frac{P}{N}}$$

$$= \frac{A\left(1-\frac{P}{N}\right) + BP}{P\left(1-\frac{P}{N}\right)}$$

comparing numerators

$$1 = A - \frac{AP}{N} + BP \Rightarrow A = 1, \quad B - \frac{A}{N} = 0 \Rightarrow B = \frac{A}{N}$$

$$= A + \left(B - \frac{A}{N}\right)P \qquad B = \frac{1}{N}$$

$$\int \frac{1}{P(1-\frac{P}{N})} dP = \int \frac{1}{P} + \frac{1}{N\left(1-\frac{P}{N}\right)} dP = \int \frac{1}{P} + \frac{1}{N-P} dP$$

$$= \ln|P| - \ln|N-P| = \ln\left|\frac{P}{N-P}\right|$$

$$\ln\left|\frac{P}{N-P}\right| = rt + C$$

$$\left|\frac{P}{N-P}\right| = e^{rt+c} = e^{rt}\cdot e^{c}$$

$$\frac{P}{N-P} = Ce^{rt}, \quad \text{where} \quad C = \pm e^{c}$$

To find $\lim_{t\to\infty} P(t)$, notice that $\lim_{t\to\infty} Ce^{rt} \to \pm\infty$ depending on the sign of C.

The left hand needs to satisfy this as well and this happen if $P\to N$ i.e the denominator $\to 0$

so $\lim_{t\to\infty} P(t) = N$.