# Homework #3

Newton's Law $\frac{dT}{dt} \propto (A - T) \Rightarrow \frac{dT}{dt} = k(A-T)$ where

$A$ is the ambient temp and $k$ is a constant.

so $\frac{dT}{dt} = k(40 - T)$

Solving: $\frac{dT}{40-T} = k\,dt \Rightarrow \int \frac{dT}{40-T} = k\int dt$

$$-\ln|40-T| = kt + c \Rightarrow \ln|40-T| = -(kt+c)$$

$$|40-T| = e^{-kt} \cdot e^{-c}$$

$$40 - T = \pm e^{-c} \cdot e^{-kt}$$

The general solution becomes

$$40 - T = Ce^{-kt}, \quad \text{where } C = \pm e^{-c}$$

we need $C$ and $K$.

$T(0) = 88$ and $T(1) = 86$

$\hookrightarrow \; 40 - 88 = Ce^{0} \Rightarrow C = -48$

$T(1) = 86$ $\quad \therefore t = 1$

$$40 - 86 = -48e^{-kt}$$

$$\frac{-46}{-48} = e^{-k \cdot 1}$$

$$k = -\ln\left(\frac{46}{48}\right) = 0.0426$$

$$40 - T = -48e^{-0.0426t} \Rightarrow T(t) = 40 + 48e^{-0.0426t}$$

$T(t) = 98.6$ at time of death so

$$98.6 = 40 + 48e^{-0.0426t}$$

$$\frac{58.6}{48} = e^{-0.0426t}$$

$$t = \frac{\ln\left(\frac{58.6}{48}\right)}{-0.0426} = -4.6839$$

which is $\cong$ 4hrs and 41 mins from 7AM also known as

$\cong$ 2:19AM

Solution #2

$$\frac{dQ}{dt} = \text{Rate in} - \text{Rate out.}$$

uniformly mixed concentration.

concentration

$$\underbrace{(2.5\,\ell/min)}_{\text{inflow}}(3g/\ell) - \underbrace{(2\,\ell/min)}_{\text{outflow}}\left(\frac{Q(t)}{vol}\,g/\ell\right)$$

$$\frac{dQ}{dt} = 7.5 - \frac{2Q}{150+\frac{t}{2}}$$

Note that in this case the volume starts at 150ℓ but gains $\frac{1}{2}\ell$ per min because of the inflow/outflow difference.

$$= 7.5 - \frac{2Q}{\frac{300+t}{2}}$$

$$= 7.5 - \frac{4Q}{300+t}$$

The initial value problem is

$$\boxed{\frac{dQ}{dt} = 7.5 - \frac{4Q}{300+t} \qquad Q(0) = 20}$$

* The IVP is linear BUT not separable

** Recal the standard form

$$\frac{dQ}{dt} + p(t)Q = g(t)$$

Integrating factor $e^{\int p(t)dt}$

*** $p(t) = \frac{4}{300+t}$ $\quad g(t) = 7.5.$

$$\frac{dQ}{dt} + \frac{4}{300+t}Q = 7.5$$

Integrating factor is $\mu(t) = e^{\int \frac{4}{300+t}dt}$

$$= e^{4\ln(300+t)} = e^{\ln(300+t)^4} = (300+t)^4$$

$$\cancel{(300+t)^4\frac{dQ}{dt} = 7.5 -}$$

$$\left(\frac{dQ}{dt} + \frac{4}{300+t}Q\right)(300+t)^4 = 7.5(300+t)^4$$

$$\frac{d}{dt}\left((300+t)^4 Q\right) = 7.5(300+t)^4$$

$$\int \frac{d}{dt}\left[(300+t)^4 Q\right] = 7.5\int (300+t)^4 dt$$

$$(300+t)^4 Q = 7.5 \cdot \frac{(300+t)^5}{5} + C$$

$$Q(t) = 7.5 \cdot \frac{(300+t)}{5} + \frac{c}{(300+t)^4}$$

$Q(0) = 20 \Rightarrow$

$$20 = \frac{7.5(300)}{5} + \frac{c}{300^4} = 20 = 450 + \frac{c}{300^4}$$

$$-430 = \frac{c}{300^4} \qquad \boxed{c = -430 \cdot (300^4)}$$

$$\boxed{Q(t) = 1.5(300+t) - \frac{430 \cdot 300^4}{(300+t)^4}}$$

As in 1a, we need to find $t$ such that

$$\frac{Q(t)}{vol} = 1$$

Recall that $vol = 150 + \frac{t}{2} = \frac{300+t}{2}$ ℓ

$$\frac{Q(t)}{vol} = \frac{1.5(300+t) - \frac{430 \cdot 300^4}{(300+t)^4}}{\left(\frac{300+t}{2}\right)} = 1.5 \cdot 2 \frac{(300+t)}{300+t} - \frac{430 \cdot 300^4}{(300+t)^4} \cdot \frac{2}{300+t}$$

$$1 = 3 - \frac{860 \cdot 300^4}{(300+t)^5} \Rightarrow \frac{860 \cdot 300^4}{(300+t)^5} = 2 \Rightarrow t = \sqrt[5]{\frac{860 \cdot 300^4}{2}} - 300$$

Solving for $t$ yields $t \approx 22.4$ mins.

At this time,

$D(t) = 150 + \frac{1}{2}(22.4) = 161.2$ so the tank overflows!

#3.

Tank 1

Let $Q_1(t)$ be the quantity of salt at time $t$

$$\frac{dQ_1}{dt} = \text{Rate in} - \text{Rate out}$$

Rate in

fresh water $\Rightarrow \left(0\ g/gal\right)\cdot\left(3gal/min\right) = 0$

Rate out $= \left(\frac{Q_1}{60}\ g/gal\right)\cdot\left(3gal/min\right)$
$= \frac{3Q_1}{60} = \frac{Q_1}{20}\ g/min$

$$\frac{dQ_1}{dt} = 0 - \frac{Q_1}{20}, \quad Q_1(0) = q_0.$$

$$\frac{dQ_1}{dt} = -\frac{Q_1}{20} \Rightarrow \int\frac{dQ_1}{Q_1} = \int -\frac{1}{20}\,dt \Rightarrow \ln|Q_1| = -\frac{1}{20}t + c$$

$$|Q_1| = e^{-\frac{1}{20}t}\cdot e^{c}$$

As $Q_1 > 0$, $|Q_1| = Q_1$ so

$Q_1 = Ce^{-t/20}$, $Q_1(0) = q_0 = C = q_0$ so $\boxed{Q_1(t) = q_0 e^{-\frac{t}{20}}}$

Tank 2

Let $Q_2(t)$ be the quantity of salt at time $t$

$$\frac{dQ_2}{dt} = \text{Rate in} - \text{Rate out}$$

Rate in

from Tank 1, at any time $t$ the concentration coming out of Tank 1 is

$\frac{Q_1(t)}{60} = \frac{q_0 e^{-\frac{t}{20}}}{60}$ therefore

$$\text{Rate in} = \left(\frac{1}{60}q_0 e^{-t/20}\right)\left(3gal/min\right) = \frac{q_0}{20}e^{-t/20}$$

and

$$\text{Rate out} = \left(\frac{Q_2(t)}{60}\right)\left(3gal/min\right) = \frac{1}{20}Q_2(t)\ g/min$$

so

$$\frac{dQ_2}{dt} = \frac{q_0}{20}e^{-t/20} - \frac{Q_2(t)}{20} \Rightarrow \frac{dQ_2}{dt} + \frac{1}{20}Q_2 = \frac{q_0}{20}e^{-t/20}$$

$$\mu(t) = e^{\int \frac{1}{20} dt} = e^{\frac{t}{20}}$$

$$\frac{d}{dt}\left(e^{\frac{t}{20}} Q_2(t)\right) = \left(\frac{q_0}{20} e^{-\frac{t}{20}}\right) e^{\frac{t}{20}}$$

$$= \frac{q_0}{20} e^{0}$$

$$e^{t/20} \cdot Q_2(t) = \int \frac{q_0}{20} dt \Rightarrow e^{t/20} \cdot Q_2(t) = \frac{q_0}{20} t + C.$$

$Q_2(0) = 0$ since tank has pure water so $C = 0$.

$$Q_2(t) = \left(\frac{q_0 t}{20}\right) e^{-t/20}$$

To find when tank 2 is saltiest. Find $Q_2'(t)$

$$Q_2'(t) = \frac{q_0}{20} \cdot e^{-t/20} + \frac{q_0 t}{20}\left(-\frac{t}{20} \cdot e^{-t/20}\right)$$

$$= \frac{q_0}{20} e^{-t/20}\left(1 + \left(-\frac{t}{20}\right)\right) = 0 \Rightarrow 1 - \frac{t}{20} = 0 \quad \underline{t = 20 \text{ mins}}$$

Check $Q_2(t)$ is concave down!

$$Q_2(20) = q_0 \cdot \frac{\cancel{20}}{\cancel{20}} \cdot e^{-20/20} = \frac{q_0}{e}$$

So Tank 2 is $\frac{1}{e}$ times salty compared to original mixture!

4. (a) $y' = \left(\frac{1}{2} + \sin(t)\right)\frac{y}{5}$

Notice that the rhs is separable.

$$\frac{dy}{dt} = \left(\frac{1}{2} + \sin(t)\right)\cdot\frac{y}{5} \Rightarrow \int \frac{5}{y}\,dy = \int \frac{1}{2} + \sin(t)\,dt$$

Integrating both sides

$$5\ln|y| = \frac{1}{2}t + (-\cos(t)) + c \quad , \quad y(0) = 1 \Rightarrow 5\ln(1) = \frac{0}{2} - \cos(0) + c$$

As $y > 0$ $\qquad \Rightarrow c = 1.$

$$5\ln(y) = \frac{t}{2} - \cos(t) + 1$$

$$\ln(y) = \frac{t}{10} - \frac{1}{5}\cos(t) + \frac{1}{5}$$

$$y = e^{\frac{t}{10} - \frac{1}{5}\cos(t) + \frac{1}{5}}$$

$$y(t) = 2 \Rightarrow \ln(2) = \frac{t}{10} - \frac{1}{5}\cos(t) + \frac{1}{5}$$

Solve using fzero in MATLAB $\qquad$ log $\equiv$ ln in MATLAB

>> $f = @(t)\left(\frac{t}{10} - \frac{1}{5}\cos(t) + \frac{1}{5} - \log(2)\right)$

>> fzero(f,0)

Yields $\underline{2.9632} \cong T.$

(b) The growth rate is $\frac{1}{10}$ so $\frac{dy}{dt} = \frac{1}{10}y$, the solution is $y(t) = y_0 e^{\frac{t}{10}}$

To find the doubling time

$$2y_0 = y_0 e^{t/10} \Rightarrow 2 = e^{t/10} \qquad t = \ln(2)\cdot 10$$
$$\cong \underline{6.9315}$$

(c) $\frac{dy}{dt} = (0.5 + \sin(2\pi t))\cdot\frac{y}{5}$, as in (a), the ODE is separable

$$\int \frac{dy}{y} = \frac{1}{5}\int (0.5 + \sin(2\pi t))\,dt$$

$$\ln(y) = \frac{1}{5}\int 0.5 + \sin(2\pi t)\,dt = \frac{1}{5}\left(0.5t - \frac{1}{2\pi}\cos(2\pi t) + c\right)$$

$$y(0) = 1 \Rightarrow \ln(1) = \frac{1}{5}\left(0 - \frac{1}{2\pi}\cos(0) + c\right) \Rightarrow 0 = \frac{-1}{10\pi} + c \qquad c = \frac{1}{10\pi}$$

$$\ln(y) = \frac{1}{5}\left(\frac{1}{2}t - \frac{1}{2\pi}\cos(2\pi t) + \frac{1}{10\pi}\right)$$

$$y(t) = e^{\frac{1}{5}\left(\frac{t}{2} - \frac{1}{2\pi}\cos(2\pi t) + \frac{1}{10\pi}\right)}$$

To find the doubling time, we need $t$ such that $y(t) = 2$

$$\ln(2) = \frac{1}{5}\left(\frac{1}{2}t - \frac{1}{2\pi}\cos(2\pi t) + \frac{1}{10\pi}\right)$$

define (MATLAB)

$$\gg f = @(t)\left(\frac{1}{5}\left(\frac{1}{2}t - \frac{1}{2\pi}\cos(2\pi t) + \frac{1}{10\pi}\right) - \log(2)\right)$$

$$\gg \text{fzero}(f, 0)$$

yeilds $\underline{6.2677}$

This is close to our solution from (b).