Homework #4

1a) In standard form

$$y' + \frac{1}{t^2} y = \frac{\sin(t)}{y^2}, \quad y(1) = 0$$

Since $\frac{1}{t^2}$ and $\frac{\sin(t)}{t^2}$ are continuous on $(-\infty, 0) \cup (0, \infty)$, the IVP has a unique solution on $(0, \infty)$.

1b) $\frac{dy}{dx} = \frac{x^2+x+1}{y-1}$, $y(0) = -1$

$f(x,y) = \frac{x^2+x-1}{y-1}$, $\quad \frac{\partial f}{\partial y} = (x^2+x+1)\left(-1 \cdot (y-1)^{-2}\right)$

so both f and $\frac{\partial f}{\partial y}$ are continuous everywhere except for $y = 1$. Our initial condition is @ $x=0$, $y=-1$ so there exists a solution to the ODE in some interval containing $(0,-1)$.

1c) $y' = \sqrt{y}$

$f(x,y) = \sqrt{y}$ $\quad \frac{\partial f}{\partial y} = \frac{1}{2} y^{-1/2}$

$\frac{\partial f}{\partial y}$ is discontinuous at $(0,0)$ so there is no unique solution.

2. First write this in the standard Bernoulli form

$$y' = xy^{-2} - 2y \Rightarrow \boxed{y' + 2y = xy^{-2}} \qquad n = -2$$

Let $v = y^{1-n} \Rightarrow v = y^3$. $\quad \frac{dv}{dx} = 3y^2 \frac{dy}{dx} \Rightarrow \frac{1}{3} y^{-2} = \frac{dy}{dx}$

Plug into ode

$$\frac{1}{3} y^{-2} \frac{dv}{dx} + 2y = xy^{-2} \Rightarrow \boxed{\frac{dv}{dx} + 6y^3 = 3x} \quad \text{a linear ODE}$$

multiply by $3y^2$

$v = y^3$

$$= \boxed{\frac{dv}{dx} + 6v = 3x}$$

$$\mu(x) = e^{\int 6\,dx} = e^{6x} \quad \text{so}$$

$$\frac{d}{dx}\left(e^{6x}\cdot v\right) = 3xe^{6x}$$

$$e^{6x}\cdot v = 3\int xe^{6x}\,dx$$

Evaluating integral by parts

$$\int xe^{6x}\,dx \underset{\uparrow}{=} \frac{x}{6}e^{6x} - \int \frac{e^{6x}}{6}\cdot dx = \frac{x}{6}e^{6x} - \frac{e^{6x}}{36} + c$$

$$u = x \qquad dv = e^{6x}$$

$$\frac{du}{dx} = 1 \qquad v = \frac{e^{6x}}{6}$$

This means

$$\left(e^{6x}\cdot v\right) = 3\left(\frac{x}{6}e^{6x} - \frac{e^{6x}}{36} + C\right) \quad \text{so that}$$

$$v = 3e^{-6x}\left(\frac{x}{6}e^{6x} - \frac{e^{6x}}{36} + C\right)$$

$$= \frac{x}{2} - \frac{1}{12} + 3Ce^{-6x}$$

and recalling that $y^3 = v$ $\quad y = v^{1/3}$ $\quad$ so $\quad y = \left(\frac{x}{2} - \frac{1}{12} + 3Ce^{-6x}\right)^{1/3}$

3. $\dfrac{dy}{dt} + Q(t)y = t, \qquad Q(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 3 & t > 2 \end{cases}$

Solve the ODE on $[0,2]$ with $y(0) = 1$, ie

$$y_1' + y_1 = t \quad , \quad y_1(0) = 1$$

$$\mu(t) = e^{\int 1\,dt} = e^t \Rightarrow \frac{d}{dt}\left(y_1 e^t\right) = te^t$$

Integrating $\quad y_1 e^t = te^t - e^t + C \quad$ and as $y_1(0) = 1$

$$1e^0 = 0e^0 - e^0 + C \Rightarrow 1 = -1 + C \quad \text{so} \quad C = 2$$

$$y_1(t) = e^{-t}\left(te^t - e^t + 2\right)$$

$$= \underline{t - 1 + 2e^{-t}}$$

on $(2, \infty)$

$$\frac{dy_2}{dt} + 3y_2 = t$$

$$\mu(t) = e^{\int 3dt} = e^{3t} \quad \text{so that}$$

$$\frac{d}{dt}\left(y_2 e^{3t}\right) = e^{3t} \cdot t$$

$$y_2 e^{3t} = \int te^{3t}\,dt = \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} + C$$

$$\Rightarrow y_2(t) = \frac{t}{3} - \frac{1}{9} + ce^{-3t}$$

We want the solution to be continuous at $t=2$ so we need

$$y_1(2) = y_2(2)$$

$$2 - 1 + 2e^{-2} = \frac{2}{3} - \frac{1}{9} + ce^{-6}$$

$$1 + 2e^{-2} = \frac{5}{9} + ce^{-6}$$

$$\frac{4}{9} + 2e^{-2} = ce^{-6} \Rightarrow c = e^6\left(\frac{4}{9} + 2e^{-2}\right)$$

so

$$y(t) = \begin{cases} t - 1 + 2e^{-t}, & 0 \le t \le 2 \\ \frac{t}{3} - \frac{1}{9} + \left[e^6\left(\frac{4}{9} + 2e^{-2}\right)\right]e^{-3t}, & t > 2. \end{cases}$$

4.

(a) If $H = 0$

$\frac{dF}{dt} = 0.5(4-F)F$ so the long term behavior of the solution is given by

the equilibrium solution(s)

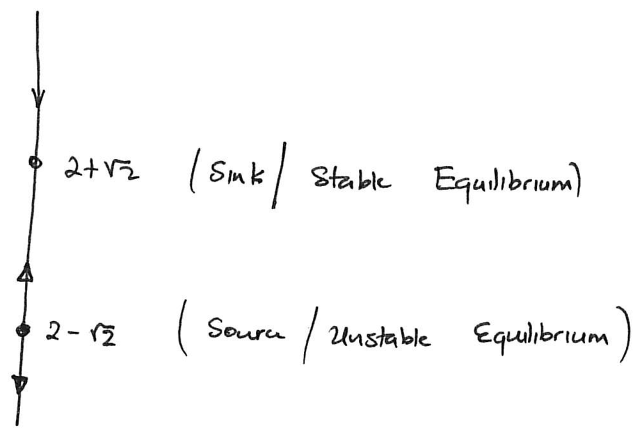
$$0.5F(4-F) = 0 \quad \Rightarrow \quad F = 0 \ , \ F = 4$$

So the long term capacity is 4000 fish.

(b) $\frac{dF}{dt} = 0.5(4-F)\cdot F - 1$

$0.5F(4-F) - 1 = 0 \quad \Rightarrow \quad 2F - 0.5F^2 - 1 = 0$. solving yields $F = 2 \pm \sqrt{2}$

Phase line



If $F(0) = 0.5 < 2-\sqrt{2}$, $F'(t) < 0$ so the population eventually becomes ~~extinc~~ extinct.

(d). $\frac{dF}{dt} = 0.5F(4-F) - H$

$= 2F - 0.5F^2 - H. \quad = -0.5F^2 + 2F - H.$

We can have either 2 equilibrium states (when quadratic has 2 roots) or 1 repeated root or no roots

The Bifurcation occurs when the discriminant $b^2 - 4ac = 0$

$$4 - (4\cdot(-0.5)\cdot -H) = 0$$

$4 + 2H = 0 \quad \Rightarrow \quad H = 2$ so the critical harvesting value is $h^* = 2000/\text{yr}$.

5. $\left(\frac{y}{x} + 6x\right) + \left(\ln(x) - 2\right)\frac{dy}{dx} = 0$

First write the ODE in the form

$M dx + N dy = 0$

$$\left(\frac{y}{x} + 6x\right) = -\left(\ln(x) - 2\right)\frac{dy}{dx}$$

$$\left(\frac{y}{x} + 6x\right)dx + \left(\ln(x) - 2\right)dy = 0$$

① Is the Equation exact?

Check $M_y = N_x$?

$M_y = \frac{1}{x}$ $\quad N_x = \frac{1}{x}$ $\quad$ so Yes!

② Find $\psi(x,y)$

$\psi_x(x,y) = \frac{y}{x} + 6x \Rightarrow \psi(x,y) = \int\left(\frac{y}{x} + 6x\right)dx$

$= y\ln|x| + \frac{6x^2}{2} + h(y)$

$\psi_y(x,y) = \frac{\partial}{\partial y}\left(y\ln(x) + 3x^2 + h(y)\right)$

$= \ln(x) + h'(y) = \ln(x) - 2$

So $h'(y) = -2 \Rightarrow h(y) = -2y + C.$

$\psi(x,y) = y\ln(x) + 3x^2 - 2y$

So our solution is $\psi(x,y) = C$ i.e $\boxed{y\ln(x) + 3x^2 - 2y = C}$

#6.

Look in your NOTES!