Homework #5 Solutions

1. $y'' + 4y' + 3y = 0$, $y(0) = 2$, $y'(0) = -1$

The characteristic polynomial is $r^2 + 4r + 3 = 0 \Leftrightarrow (r+1)(r+3) = 0 \Rightarrow r = -1, -3.$

Hence $y(t) = c_1 e^{-t} + c_2 e^{-3t}$

Applying initial conditions, $y(0) = 2 \Rightarrow c_1 + c_2 = 2 \quad \ldots (i)$

$y'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$, therefore $y'(0) = -1 \Rightarrow -c_1 - 3c_2 = -1. \quad \ldots (ii)$

From (i) $c_1 = 2 - c_2 \Rightarrow$ ~~-c2 2~~ $-(2 - c_2) - 3c_2 = -1$

$-2 + c_2 - 3c_2 = -1 \Rightarrow -2 - 2c_2 = -1$

$-2c_2 = 1 \Rightarrow c_2 = \frac{-1}{2}.$

$C_1 = 2 - C_2 = 2 - \left(-\frac{1}{2}\right) = \frac{5}{2}$

$y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t}$

$\lim_{t \to \infty} y(t) = 0$

2. $y'' + 8y' - 9y = 0$

$y(0) = \alpha$, $y'(0) = 1.$

$r^2 + 8r - 9 = 0 \Leftrightarrow (r-1)(r+9) = 0 \Rightarrow r = +1, -9$

$y(t) = c_1 e^{t} + c_2 e^{-9t}$ $\qquad y'(t) = c_1 e^{t} - 9c_2 e^{-9t}$

$y(0) = \alpha \Rightarrow C_1 + C_2 = \alpha \quad \ldots (i)$

$y'(0) = 1 \Rightarrow C_1 - 9C_2 = 1. \quad \ldots (ii)$

Solving, $C_1 = \alpha - C_2 \Rightarrow (\alpha - C_2) - 9C_2 = 1$

$\alpha - 10C_2 = 1 \Rightarrow \frac{\alpha - 1}{10} = C_2$

$C_1 = \alpha - C_2$

$= \alpha - \left(\frac{\alpha - 1}{10}\right)$

$C_1 = \frac{9\alpha}{10} + \frac{1}{10} = \frac{9\alpha+1}{10}.$

For $y(t) \to 0$ as $t \to \infty$, we need $C_1 = 0 \Rightarrow \frac{9\alpha+1}{10} = 0$ $\quad \underline{\alpha = -\frac{1}{9}}$

3. $3y'' + y' + 2y = 1$

If y is an equilibrium solution, $y'(t) = 0$ and $y''(t) = 0 \Rightarrow 2y = 1$, hence

$y = \frac{1}{2}.$

4. Write the IVP in standard form as

$$y'' + \frac{1}{x-2} y' + \tan(x) y = 0$$

The coefficient functions are continuous for all x such that $x \neq 2, \frac{\pi}{2} \pm k\pi$.

As $x_0 = 3$, the IVP will have a unique solution on $(2, \frac{3\pi}{2})$.

5. To show that $y_1(t) = t$ and $y_2(t) = te^t$ are solutions plug into ODE

$y_1(t) = t$ $\qquad y_2(t) = te^t$

$y_1'(t) = 1$ $\qquad y_2'(t) = te^t + e^t \cdot 1$

$y_1''(t) = 0$ $\qquad y_2''(t) = te^t + e^t + e^t = te^t + 2e^t.$

$$t \underbrace{y_1''(t)}_{0} - (t(t+2)) \cdot 1 + (t+2) \cdot t = -t^2 - 2t + t^2 + 2t = 0 \quad ✓$$

$$t^2 \left[te^t + 2e^t \right] - (t^2+2t)\left[te^t + e^t \right] + (t+2)\left[te^t \right]$$

$$\cancel{t^3 e^t} + 2t^2 e^t - \cancel{t^3 e^t} - 2t^2 e^t - t^2 e^t - 2te^t + t^2 e^t + 2te^t = 0 \quad ✓$$

to show that y_1 and y_2 are a fundamental set, compute $W[y_1, y_2] = y_1 y_2' - y_1' y_2$

$W[y_1, y_2](t) = t \cdot (te^t)' - (t)'(te^t)$

$= t(te^t + e^t) - 1 \cdot (te^t)$

$= t^2 e^t + te^t - te^t = t^2 e^t > 0$ because $t > 0$.

* Note the ode is $t^2 y'' - t(t+2)y' + (t+2)y = 0$, $t > 0$ This was missing on the problem

6.

$ty'' + 2y' + te^t y = 0$ in standard form we have

$y'' + \frac{2}{t} y' + e^t y = 0.$

Recall that if $y'' + p(t)y' + q(t)y = 0$, then $W[y_1, y_2] = ce^{-\int p(t)dt}$

$= ce^{-\int \frac{2}{t} dt}$

$= ce^{-2\ln|t|}$

$W[y_1, y_2](t) = ce^{-2\ln|t|} = ce^{\ln t^{-2}} = \frac{c}{t^2}$

We are given that $W[y_1, y_2](1) = 3 \Rightarrow \frac{c}{1^2} = 3 \Rightarrow c = 3.$

$W[y_1, y_2](5) = \frac{3}{t^2} = \frac{3}{25}.$