Homework #6 SOLUTIONS

1 (a) $y'' + 6y' + 9y = 0 \Rightarrow r^2 + 6r + 9 = 0 \Leftrightarrow (r+3)^2 = 0 \Rightarrow r = -3$, twice

so $y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$.

1 (b) $y'' + 2y' + 5y = 0 \Rightarrow r^2 + 2r + 5 = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{2^2 - (4 \cdot 1 \cdot 5)}}{2}$

$r = \dfrac{-2 \pm \sqrt{-16}}{2} \Rightarrow -1 \pm 2i$

$$\boxed{y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)}$$

2. (a) $y'' + 2y' + 2y = 0$, $y\left(\frac{\pi}{4}\right) = 2$, $y'\left(\frac{\pi}{4}\right) = -2$

$r^2 + 2r + 2 = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{2^2 - (4 \cdot 1 \cdot 2)}}{2} = -1 \pm i$

$y(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$

$y'(t) = c_1 e^{-t}(-\cos t - \sin t) + c_2 e^{-t}(-\sin(t) + \cos(t))$

$y\left(\frac{\pi}{4}\right) = 2 \Rightarrow c_1 e^{-\frac{\pi}{4}} \frac{\sqrt{2}}{2} + c_2 e^{-\pi/4} \cdot \frac{\sqrt{2}}{2} = 2$

$y'\left(\frac{\pi}{4}\right) = -2 \Rightarrow -\sqrt{2} c_1 e^{-\pi/4} = -2.$

$c_1 = \sqrt{2} e^{\pi/4}$ $\quad c_2 = \sqrt{2} e^{\pi/4}$

$$\boxed{y(t) = \sqrt{2} e^{-(t - \pi/4)} \cos(t) + \sqrt{2} e^{-(t-\pi/4)} \sin(t)}$$

2 (b) $y'' + 4y' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 1$

$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$, $r = -2$, twice

$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$.

$y'(t) = -2c_1 e^{-2t} + c_2 (1 - 2t) e^{-2t}$

plug in initial conditions

$$y(-1) = 2, \quad y'(-1) = 1 \quad \Rightarrow \quad c_1 e^2 - c_2 e^2 = 2 \quad \ldots (i)$$

$$-2c_1 e^2 + 3c_2 e^2 = 1 \quad \ldots (ii)$$

$$2\times(i) + (ii) \Rightarrow \quad \begin{array}{r} 2c_1 e^2 - 2c_2 e^2 = 4 \\ -2c_1 e^2 + 3c_2 e^2 = 1 \\ \hline c_2 e^2 = 5 \end{array} \quad \Rightarrow \quad c_2 = 5e^{-2}$$

plug into (i) $\quad c_1 e^2 - (5e^{-2})e^2 = 2$

$$c_1 e^2 - 5 = 2 \Rightarrow c_1 e^2 = 7 \Rightarrow c_1 = 7e^{-2}.$$

3. $ay'' + by' = 0 \Rightarrow ar^2 + br = 0 \Rightarrow r(ar+b) = 0$

$r = 0, \; r = \frac{-b}{a} < 0$ because $a, b > 0$.

therefore

$$y(t) = c_1 e^{0t} + c_2 e^{-\frac{b}{a}t}$$

$$= c_1 + c_2 e^{-\frac{b}{a}t}$$

$$y(0) = y_0 \Rightarrow \quad y_0 = c_1 + c_2 \quad \ldots (i).$$

$$y'(t) = c_2 \cdot \left(-\frac{b}{a}\right) e^{-\frac{b}{a}t}$$

$$y'(0) = y_0' \Rightarrow c_2\left(-\frac{b}{a}\right)\underbrace{e^{-\frac{b}{a}\cdot 0}}_{1} = y_0' \Rightarrow c_2 = y_0'\left(-\frac{a}{b}\right)$$

Plug into (i)

$$y_0 = c_1 + c_2 \Rightarrow c_1 = y_0 - c_2 \Rightarrow c_1 = y_0 + y_0'\left(\frac{a}{b}\right)$$

We conclude that $\quad y(t) = \left(y_0 + \frac{y_0' a}{b}\right) - \frac{y_0' a}{b} e^{-\frac{b}{a}t}$

therefore $\quad \lim_{t \to \infty} (y(t)) = y_0 + \frac{y_0' a}{b}$