Homework #7

1.a $y'' - y' - 2y = -2t + 4t^2$

Step 1

Solve the homogeneous equation,

$$y'' - y' - 2y = 0,$$

the characteristic equation is $r^2 - r - 2 = 0 \Leftrightarrow (r+1)(r-2) = 0 \Rightarrow r = -1, 2$ therefore

$$y_c(t) = c_1 e^{-t} + c_2 e^{2t}$$

Step II

Our rhs is a polynomial and $r=0$ is not a root of the characteristic polynomial therefore

$$y_p(t) = At^2 + Bt + C.$$

Plug into ODE

$$y_p'(t) = 2At + B, \quad y_p''(t) = 2A.$$

therefore

$$y_p'' - y_p' - 2y_p = 2A - (2At + B) - 2(At^2 + Bt + C)$$

$$= (2A - B - 2C) - 2At - 2Bt - 2At^2$$

$$= (2A - B - 2C) - (2A + 2B)t - 2At^2 = -2t + 4t^2$$

Comparing coefficients

t^2 $\quad -2A = 4 \Rightarrow \underline{A = -2}$

t $\quad -(2A + 2B) = -2 \Rightarrow -(2 \cdot (-2) + 2B) = -2$

$-(-4 + 2B) = -2 \Rightarrow -4 + 2B = 2$

$2B = 6 \Rightarrow \underline{B = 3}$

constants

$2A - B - 2C = 0.$

$2(-2) - 3 - 2C = 0$

$-4 - 3 = 2C \Rightarrow 2C = -7 \Rightarrow C = -\frac{7}{2}.$

$$\boxed{y_p(t) = -2t^2 + 3t - \frac{7}{2}}$$

plug into ODE

$$y_p'' + 2y_p' = \left[-4A\sin(2t) - 4B\cos(2t)\right] + 2\left[2A\cos(2t) - 2B\sin(2t)\right]$$

$$= \left[-4A - 4B\right]\sin(2t) + \left[-4B + 4A\right]\cos(2t) = 8\sin(2t) + 24\cos(2t)$$

Comparing coefficients

$-4A - 4B = 8 \quad \ldots (i)$

$-4B + 4A = 24 \quad \ldots (ii)$

$(i) + (ii) \Rightarrow \quad -8B = 32$

$B = \frac{-32}{8} = -4$

plug into (i) $\quad -4A = 8 + 4B$

$-4A = 8 + 4(-4)$

$= 8 - 16 = -8 \qquad A = 2$

$$\boxed{y_p(t) = 2\sin(2t) - 4\cos(2t)}$$

Extra Note

If we had initial conditions, say $y(0)=2$, $y'(0)=3$ ie the problem reads

Solve (IVP)

$$y'' + 2y' = 8\sin(2t) + 24\cos(2t)$$
$$y(0) = 2, \quad y'(0) = 3$$

$$y(t) = y_c(t) + y_p(t) = c_1 + c_2 e^{-2t} + 2\sin(2t) - 4\cos(2t)$$

then use the initial conditions to solve for c_1 and c_2. Try it!

(* Several * mistakes on homework)

You should get

$$\boxed{y(t) = \frac{11}{2} + \frac{1}{2}e^{-2t} + 2\sin(2t) - 4\cos(2t)}$$

Some of you found c_1 and c_2 before y_p.

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$$y'' - 4y' + 4y = 36e^{4t}$$
$$y(0) = 7, \quad y'(0) = 7$$

$r^2 - 4r + 4 = 0 \Leftrightarrow (r-2)^2 = 0$, $r = 2$, twice, therefore

$$y_c(t) = c_1 e^{2t} + c_2 t e^{2t}$$

In this case $r=4$ is NOT a root of the characteristic polynomial so we can choose

$$y_p(t) = Ae^{4t}$$ (ie No need to multiply by any factors of t^s!)

$y_p(t) = Ae^{4t}$

$y_p'(t) = 4Ae^{4t}$, $y_p''(t) = 16Ae^{4t}$

$y_p'' - 4y_p' + 4y_p = 16Ae^{4t} - 4(4Ae^{4t}) + 4Ae^{4t} = 36e^{4t}$

$16Ae^{4t} - 16Ae^{4t} + 4Ae^{4t} = 36e^{4t} \Rightarrow 4Ae^{4t} = 36e^{4t}$

$4A = 36 \Rightarrow A = \frac{36}{4} = 9$

$y_p(t) = 9e^{4t}$, therefore

$$\boxed{y(t) = c_1e^{2t} + c_2te^{2t} + 9e^{4t}}$$

Our initial conditions are $y(0) = 7$, $y'(0) = 7$ $\Rightarrow$

$c_1e^0 + c_2 \cdot 0e^0 + 9 = 7$.

$c_1 + 9 = 7 \Rightarrow c_1 = 7 - 9 = -2$

$y'(t) = 2c_1e^{2t} + c_2[t \cdot 2e^{2t} + e^{2t} \cdot 1] + 36e^{4t}$

$= -4e^{2t} + c_2[t \cdot 2e^{2t} + e^{2t}] + 36e^{4t}$

$y'(0) = 7$

↳ $-4 + c_2[1] + 36 = 7 \Rightarrow c_2 = -25$

therefore

$$\boxed{y(t) = -2e^{2t} - 25te^{2t} + 9e^{4t}}$$

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$y'' + 2y' + 5y = e^{-t}\cos(2t)$

$y(0) = 1$, $y'(0) = 1$

The characteristic equation is $r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{2^2 - (4 \cdot 1 \cdot 5)}}{2} = \frac{-2 \pm \sqrt{4 - 20}}{2}$

$= \frac{-2 \pm i\sqrt{16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$y_c(t) = c_1e^{\lambda t}\cos(\mu t) + c_2e^{\lambda t}\sin(\mu t)$

$y_c(t) = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t)$

In this case, our rhs is $e^{-t}\cos(2t)$ and $-1+2i$ is a root of the characteristic polynomial so choose

$$y_p(t) = Ate^{-t}\cos(2t) + Bte^{-t}\sin(2t)$$

3. (a) $y'' + 3y' + 2y = e^{t}(t^2+1)\sin(2t)$

The characteristic polynomial for the homogeneous problem is

$$r^2 + 3r + 2 = 0 \iff (r+1)(r+2) = 0 \Rightarrow r = -1, -2.$$

$$y_c(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

$r = 1 + 2i$ is not a root of the characteristic polynomial [case (c)] of summary, therefore

$$\boxed{y_p(t) = (A_0 + A_1 t + A_2 t^2)e^{t}\cos(2t) + (B_0 + B_1 t + B_2 t^2)e^{t}\sin(2t).}$$

(b) $y'' + 6y' + 9y = (t+1)e^{-3t} + (2t+1).$

The characteristic polynomial for the homogeneous problem is

$$r^2 + 6r + 9 = 0 \iff (r+3)^2 = 0 \qquad r = -3, \text{ twice}.$$

$$y_c(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

We break up the sum into 2 problems

(i) $y'' + 6y' + 9y = (t+1)e^{-3t}$

$r = -3$ is a double root of the characteristic polynomial, therefore

$$y_p(t) = t^2(A_0 + A_1 t)e^{-3t}$$

(ii) $y'' + 6y' + 9y = (2t+1)$

$r = 0$ is not a root of the characteristic polynomial, therefore

$$y_p(t) = B_0 + B_1 t$$

Combining the 2 particular solutions

$$\boxed{y_p(t) = t^2(A_0 + A_1 t)e^{-3t} + (B_0 + B_1 t).}$$