Homework # 8 solutions

1.

$$y'' + y' = \frac{1}{1+e^{-t}}, \quad \text{given } y_1(t) = e^{-t}, \; y_2(t) = 1.$$

Variation of parameters

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t) \quad \text{where}$$

$$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W[y_1, y_2](t)}\, dt \qquad u_2(t) = \int \frac{y_1(t)\, g(t)}{W[y_1, y_2](t)}\, dt$$

Our ode is in the standard form, therefore $g(t) = \frac{1}{1+e^{-t}}$, $W[e^{-t}, 1] = e^{-t} \cdot \frac{d}{dt}(1) - \frac{d}{dt}(e^{-t}) \cdot 1$

$$= -(-1 \cdot e^{-t})$$
$$= e^{-t} \neq 0$$

$$u_1(t) = -\int \frac{1 \cdot 1}{1+e^{-t}} \cdot \frac{1}{e^{-t}}\, dt = -\int \frac{e^t}{1+e^{-t}}\, dt$$

Simplify integrand

$$\frac{e^t}{1+e^{-t}} = \frac{e^t}{1+\frac{1}{e^t}} = \frac{e^t}{\frac{e^t+1}{e^t}} = \frac{e^t \cdot e^t}{e^t+1}$$

$$-\int \frac{e^t \cdot e^t}{e^t+1} = -\int \frac{(u-1)\, du}{u} = -\int 1 - \frac{1}{u}\, du = \int \frac{1}{u} - 1\, du = \ln|u| - u$$

$$= \ln(e^t+1) - (e^t+1)$$

Let $u = e^t + 1$

$\frac{du}{dt} = e^t \Rightarrow du = e^t \cdot dt$

$u - 1 = e^t$.

$$u_1(t) = \ln(e^t+1) - (e^t+1)$$

$$u_2(t) = \int e^{-t} \cdot \frac{1}{1+e^{-t}} \cdot \frac{1}{e^{-t}}\, dt = \int \frac{e^{-t} \cdot e^t}{1+e^{-t}}\, dt = \int \frac{1}{1+e^{-t}}\, dt$$

Simplify integrand $\quad \frac{1}{1+e^{-t}} = \frac{1}{1+\frac{1}{e^t}} = \frac{1}{\frac{e^t+1}{e^t}} = \frac{e^t}{e^t+1}$

$$u_2(t) = \int \frac{e^t}{e^t+1}\, dt = \int \frac{1}{u}\, du = \ln|u| = \ln(e^t+1)$$

$$y_p(t) = \left[\ln(e^t+1) - (e^t+1)\right] e^{-t} + \left[\ln(e^t+1)\right]\cdot 1$$

The general solution is

$$\boxed{y(t) = c_1 e^{-t} + c_2 \cdot 1 + \left[\ln(e^t+1) - (e^t+1)\right] e^{-t} + \left[\ln(e^t+1)\right].}$$

2. $$y'' + y = \tan(t) + 3t - 1.$$

Break up into

$$y_1'' + y_1 = \tan(t) \quad A$$

$$y_2'' + y_2 = (3t-1)$$

Step 1

Solve homogeneous problem

$$y'' + y = 0 \Rightarrow r^2 + 1 = 0 \quad r = \pm\sqrt{-1} = \pm i$$

$$y_c(t) = e^{0t}\cos(t) + e^{0t}\sin(t) = \cos(t) + \sin(t).$$

A Variation of parameters

$$y_1^p(t) = u_1(t)y_1(t) + u_2(t)y_2(t). \quad \text{where } y_1(t) = \cos(t)$$
$$y_2(t) = \sin(t)$$

$$W[\cos(t), \sin(t)] = \cos(t)\cdot\cos(t) - \sin(t)\cdot(-\sin(t)) = \cos^2(t) + \sin^2(t) \equiv 1.$$

$$u_1(t) = -\int \frac{\sin(t)\cdot\tan(t)}{1}\,dt$$

$$= -\int \sin(t)\cdot\frac{\sin(t)}{\cos(t)}\,dt = -\int\frac{\sin^2(t)}{\cos(t)}\,dt = \int\frac{-(1-\cos^2(t))}{\cos(t)}\,dt$$

$$= -\int\frac{1}{\cos(t)} + \int\frac{\cos^2(t)}{\cos(t)}\,dt$$

$$= -\int\sec(t)\,dt + \int\cos(t)\,dt$$

$$= -\ln\left(\sec(t) + \tan(t)\right) + \sin(t)$$

$$u_2(t) = \int \frac{\cos(t)\cdot\tan(t)}{1}\,dt = \int \cos(t)\cdot\frac{\sin(t)}{\cos(t)}\,dt = \int \sin(t)\,dt = -\cos(t)$$

$$y_1^P(t) = \left[-\ln\left(\sec(t) + \tan(t)\right) + \sin(t)\right]\cos(t) + \left[-\cos(t)\right]\sin(t).$$

for (B) use method of undetermined coefficients

$y_2'' + y_2 = 3t+1$, $y_c = \sin(t) + \cos(t)$ so pick

$y_2^P(t) = At + B$

$y_2^{P\prime}(t) = A$

$y_2^{P\prime\prime}(t) = 0$, plug into ODE

$y_2^{P\prime\prime} + y_2^P = (At+B) = (3t+1) \Rightarrow A = 3$, $B = 1.$

$y_2^P(t) = 3t+1.$

The general solution is

$$\cos(t) + \sin(t) + \left[-\ln\left(\sec(t) + \tan(t)\right)\right]\cos(t) - \cos(t)\sin(t) + (3t+1)$$

3. The general equation for the spring-mass system is

$$mu'' + \gamma u' + ku = 0$$

$m = 1\text{kg}$, $k = 4\,\text{N/m}$, $\gamma = 5\,\text{N/m}$ so we have

$$u'' + 5u' + 4u = 0$$

Case I

mass is lifted and released $\Rightarrow$ $u(0) = -1$ (remember $\downarrow$ is "+")

$u'(0) = 0$

Solving

$$u'' + 5u' + 4u = 0 \iff r^2 + 5r + 4 = 0 \iff (r+1)(r+4) = 0$$

$r = -1, \; r = -4$

$$u(t) = c_1 e^{-t} + c_2 e^{-4t}$$

Applying initial conditions

$u(0) = -1 \Rightarrow \quad c_1 + c_2 = -1 \quad \ldots (i)$

$u'(t) = -c_1 e^{-t} - 4c_2 e^{-4t} \Rightarrow u'(0) = 0 \Rightarrow -c_1 - 4c_2 = 0 \quad \ldots (ii)$

Solving

$c_1 = -1 - c_2$ plug into (ii) $\quad -(-1-c_2) - 4c_2 = 0$

$1 + c_2 - 4c_2 = 0$

$1 - 3c_2 = 0 \Rightarrow c_2 = \frac{1}{3}$

$c_1 = -1 - \frac{1}{3} = -\frac{4}{3}$

$$u(t) = \frac{-4}{3} e^{-t} + \frac{1}{3} e^{-4t}.$$

Case II

lifted 1 m and given a downward velocity of 4 m/s

$u(0) = -1$

$u'(0) = 4$

$u(0) = -1 \Rightarrow \cancel{c_1c} \; c_1 + c_2 = -1 \quad \ldots (i)$

$u'(0) = 4 \quad \Rightarrow -c_1 - 4c_2 = 4 \quad \ldots (ii)$

from $\quad c_1 = -1 - c_2 \quad \underset{\text{plug into (ii)}}{\Rightarrow} \quad (-1 - c_2) - 4c_2 = 4$

$-1 - 5c_2 = 4$

$-5c_2 = 5 \qquad c_2 = -1$

$c_1 = -1 - (-1) = 0$

$$u(t) = -e^{-4t}$$

4.

$$mu'' + \gamma u' + ku = 0$$

$$u(0) = u_0, \quad u'(0) = 0$$

The system is critically damped $\Rightarrow$ $b\ \gamma^2 - 4mk = 0$, therefore $r = -\frac{\gamma}{2m}$

$$u(t) = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}$$

Applying initial conditions

$$u(0) = u_0, \quad u'(0) = 0$$

$$u(0) = u_0 \Rightarrow c_1 + 0 = u_0 \Rightarrow c_1 = u_0 \Rightarrow u(t) = u_0 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}$$

$$u'(t) = u_0\left(\frac{-\gamma}{2m}\right) e^{-\frac{\gamma}{2m}t} + c_2\left[t \cdot \left(\frac{-\gamma}{2m}\right) e^{-\frac{\gamma}{2m}t} + e^{-\frac{\gamma}{2m}t} \cdot 1\right]$$

$$u'(0) = 0 \Rightarrow u_0\left(\frac{-\gamma}{2m}\right) + c_2\left[e^0\right] = 0$$

$$c_2 = \frac{u_0 \gamma}{2m}$$

$$u(t) = u_0 e^{-\frac{\gamma}{2m}t} + \frac{u_0 \gamma}{2m} t e^{-\frac{\gamma}{2m}t}$$

Let's show that !

(a) $\lim_{t \to \infty} u(t) = 0$

$$\lim_{t \to \infty}\left[u_0 e^{-\frac{\gamma}{2m}t}\right] + \lim_{t \to \infty}\left[\frac{u_0 \gamma}{2m} t e^{-\frac{\gamma}{2m}t}\right]$$

$= 0$ because $e^{-kt} \to 0$

here use l'Hopitals rule

$$\lim_{t \to \infty} t e^{-\frac{\gamma}{2m}t} = \lim_{t \to \infty} \frac{t}{e^{+\frac{\gamma}{2m}t}} = \lim_{t \to \infty} \frac{1}{\frac{+\gamma}{2m} e^{+\frac{\gamma}{2m}t}}$$

$= 0$ because $e^{kt} \to$

(b) $u(t) = 0$ for $t > 0$, assuming $u_0 \neq 0$, otherwise the ~~pendulum~~ mass goes nowhere!

Try to solve

$u(t) = 0 \Rightarrow$

$$u_0 e^{-\frac{\gamma}{2m}t} + u_0 \frac{\gamma}{2m} t e^{-\frac{\gamma}{2m}t} = 0$$

$$\Rightarrow \cancel{u_0} \frac{\gamma}{2m} t \cancel{e^{-\frac{\gamma}{2m}t}} = -\cancel{u_0} \cancel{e^{-\frac{\gamma}{2m}t}}$$

So $t = -\frac{2m}{\gamma} < 0$ because $m, \gamma > 0$ so the mass does not cross the equilibrium position for any $t > 0$.