1. (a) (6 points) For each of the following, determine whether the equation is **linear** or **non-linear** and state the **order** of the equation.

   1. $y' = \frac{y}{t} + t^2 \sin(2t)$

      $y' - \frac{y}{t} = t^2 \sin(2t)$ has the form $y' + p(t)y = g(t)$

      so LINEAR and first order

   2. $\frac{dy}{dx} - \frac{2y}{x} = x^{-1}y^{-1}$ (This is a Bernoulli type) ($n = -1$)

      Non linear and first order

   3. $y'' + \cos(t + y) = e^t$

      2nd order due to y''

      nonlinear due to $\cos(t+y)$

   (b) (6 points) So far we have studied **SEPARABLE, BERNOULLI** and **EXACT** ODEs. Classify each of the following using these 3 categories. Provide a brief justification for your answer. If none apply, simply state **NONE**.

   1. $\frac{dy}{dx} = 2xy^2 + 4y$

      $y' - 4y = 2xy^2$

      Bernoulli, $n = 2$

   2. $(x^2 + y)dx + (x - \sin(y))dy = 0$

      $\frac{\partial}{\partial y}(x^2+y) = 1 = \frac{\partial}{\partial x}(x - \sin(y)) = 1$ so this one is exact.

   3. $\frac{dy}{dx} = e^{x+y}$

      $\frac{dy}{dx} = e^x \cdot e^y$ so separable.

2. (a) (10 points) Solve

$$t\frac{dy}{dt} + 2y = \frac{t}{t^3+1}$$

$$y' + \frac{2}{t} = \frac{t}{t^3+1} \cdot \frac{1}{t}$$

$$y' + \frac{2}{t} = \frac{1}{t^3+1}$$ this equation is linear

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln(t^2)} = t^2.$$ (+2)

$$(t^2y)' = \frac{t^2}{t^3+1}$$ (+3)

$$t^2y = \int \frac{t^2}{t^3+1}dt$$

let $u = t^3+1$

$$\frac{du}{dt} = 3t^2 \Rightarrow \frac{1}{3}du = t^2dt$$

so $$\int \frac{t^2}{t^3+1}dt = \frac{1}{3}\int \frac{1}{u}du$$

$$= \frac{1}{3}\ln|u| + c$$ (+4)

$$= \frac{1}{3}\ln|t^3+1| + c$$

$$\downarrow$$

$$t^2y = \frac{1}{3}\ln|t^3+1| + c$$

(+1)

(b) (10 points) Solve

$$\frac{dy}{dt} = ty^2 - t$$

$\cancel{y(0) = -1}$ $y(-1) = 0$

+2

$\frac{dy}{dt} = t(y^2-1)$ so this one is separable.

$\int \frac{dy}{y^2-1} = \int t\,dt$.

LHS Integral

$\int \frac{1}{y^2-1}\,dy = \int \frac{1}{(y+1)(y-1)}\,dy$

Partial fraction decomposition

$\frac{1}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}$

$\frac{1}{(y+1)(y-1)} = \frac{A(y-1) + B(y+1)}{(y+1)(y-1)}$

$1 = Ay - A + By + B$

$1 = (A+B)y + (B-A)$

$B - A = 1 \ldots (i)$

$A + B = 0 \ldots (ii)$

$B = -A$, plug into (ii)

$-A - A = 1 \Rightarrow -2A = 1$

$A = \frac{-1}{2}$

$B = \frac{1}{2}$

LHS integral becomes

$\frac{1}{2}\int \frac{1}{y-1} - \frac{1}{y+1}\,dy = \frac{1}{2}\left(\ln|y-1| - \ln|y+1|\right)$

+5

$= \frac{1}{2}\ln\left(\left|\frac{y-1}{y+1}\right|\right)$

So $\frac{1}{2}\ln\left(\left|\frac{y-1}{y+1}\right|\right) = \frac{t^2}{2} + c$ +3

$y(0) = -1 \Rightarrow$

$\frac{1}{2}\ln\left|\frac{0-1}{0+1}\right| = \frac{1}{2} + c$

$\frac{1}{2}\ln 1 = \frac{1}{2} + c \qquad c = \frac{-1}{2}$

$$\boxed{\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{t^2}{2} - \frac{1}{2}}$$

(c) (10 points) Transform the equation

$$\frac{dy}{dt} = -y + e^t y^2$$

into a first order ODE. **Do not solve the resulting ODE.**

Let $v = y^{2-1} = y^{-1}$

$\frac{dv}{dt} = -y^{-2} \cdot \frac{dy}{dt} \Rightarrow -y^2 \frac{dv}{dt} = \frac{dy}{dt}$

$-y^2 \frac{dv}{dt} + y = -e^t y^2$

$\frac{dv}{dt} - y^{-1} = -e^t$ but $y^{-1} = v$ so

$\boxed{v' - v = -e^t}$

3. A 50 litre tank initially contains q_0 grams of salt. A salty mixture with a concentration $e^{-\frac{t}{25}}$ grams/litre at time t flows in at a rate of 2 litres/minute, and a throughly stirred mixture comes out at the same rate.

(a) (15 points) **Set up** and **solve** a differential equation to determine the amount of salt at time t.

Let $Q(t)$ be the quantity of salt at time t

$Q(0) = q_0$

$$\frac{dQ}{dt} = \text{Rate in} - \text{Rate out}$$

$$= \left(e^{-\frac{t}{25}}\, g/l\right)\left(2\,l/min\right) - \left(\frac{Q(t)}{50}\right)\left(2\,l/min\right)$$

$$= 2e^{-\frac{t}{25}} - \frac{Q}{25}$$

+8

$$\boxed{\frac{dQ}{dt} + \frac{Q}{25} = 2e^{-\frac{t}{25}}}$$

$$\mu(t) = e^{\int \frac{1}{25}dt} = e^{\frac{t}{25}}$$

$$\frac{d}{dt}\left(Q e^{t/25}\right) = 2e^{-t/25} \cdot e^{t/25}$$

$$\frac{d}{dt}\left(Q e^{t/25}\right) = 2 \text{, integrating}$$

$$Q e^{t/25} = 2t + C, \quad Q(0) = q_0 \Rightarrow q_0 = C$$

$$Q e^{t/25} = 2t + q_0 \Rightarrow Q(t) = (2t + q_0)e^{-t/25}$$

(b) (5 points) Suppose you discover a small leak in the tank and estimate it to be about 0.1 litres/minute. **State but do not solve** the new differential equation.

The leak changes the outflow

① New rate of outflow $= 2 + 0.1 = 2.1\, l/min$

② Volume of tank decreases, $V(t) = 50 - 0.1t$

$$\frac{dQ}{dt} = 2e^{-\frac{t}{25}} - \left(\frac{Q}{50 - 0.1t}\right)\left(2.1\, l/min\right)$$

4. Given

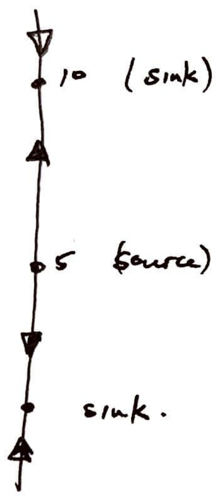
$$\frac{dy}{dt} = -r(1 - \frac{y}{5})(1 - \frac{y}{10})y$$

where $r > 0$ is a constant.

(a) (13 points) Find and classify the equilibrium solutions to the ODE. Show your equilibrium solution on a phase line.

Equilibrium points $-r(1-\frac{y}{5})(1-\frac{y}{10})y = 0 \Rightarrow (1-\frac{y}{5}) = 0$ or $(1-\frac{y}{10}) = 0$, or $y = 0$

$y=0$, $y=5$, $y=10$



To find the sign, plug in any value inside each interval into $\frac{dy}{dt}$ and find the sign

(b) (5 points) Suppose $y(0) = 6$. Determine $\lim_{t\to\infty} y(t)$.

If $y(0) = 6$, $\frac{dy}{dt} > 0$ so $y(t) \to 10$.

5. (15 points) Solve $(x^2 + y)dx + (x - \sin(y))dy = 0$.

$M(x,y) = x^2 + y$ $\quad$ $N(x,y) = x - \sin(y)$

$M_y = 1$ $\quad$ $N_x = 1$, so Exact

Let $\psi_x(x,y) = x^2 + y$

$$\psi(x,y) = \int (x^2+y)\,dx = \frac{x^3}{3} + xy + h(y)$$

$$\frac{\partial}{\partial y}\left(\frac{x^3}{3} + xy + h(y)\right) = x + h'(y) = x - \sin(y).$$

Therefore $h'(y) = -\sin(y) \Rightarrow h(y) = \cos(y)$

Our solution is

$$\psi(x,y) = \frac{x^3}{3} + xy + \cos(y) = C.$$

6. (10 points) Solve $\frac{dy}{dx} = \frac{1}{e^y + x}$

$(e^y + x)\,dy = dx \quad \Rightarrow \quad -dx + (e^y + x)dy = 0$

$M = -1$

$N = e^y + x$

Is this Exact?

$M_y = 0$ and $N_x = 1$, so NO.

Lets make it exact

$$\mu(y) = e^{\int \frac{N_x - M_y}{-1}} = e^{-\int 1-0\,dy} = e^{-y}$$

$$-e^{-y}dx + e^{-y}(e^y + x)\,dy = 0$$

$$\Rightarrow -e^{-y}dx + (1 + xe^{-y})\,dy = 0$$

$M = -e^{-y}$ $\qquad N = 1 + xe^{-y}$

$M_y = e^{-y}$ $\qquad N_x = e^{-y}$, $M_y = N_x$, Yes this is exact

Let $\psi_x(x,y) = -e^{-y}$ then $\psi(x,y) = -e^{-y}x + h(y)$

$$\frac{\partial}{\partial y}\left(-e^{-y}x + h(y)\right) = e^{-y}x + h'(y) = 1 + xe^{-y}$$

so $h'(y) = 1 \Rightarrow h(y) = y$.

Our solution,

$$\psi(x,y) = -xe^{-y} + y = c$$