1. (a) (6 points) For each of the following, determine whether the equation is \textit{linear} or \textit{nonlinear} and state the order of the equation.

1. \[ y' = \frac{y}{t} + t^2 \sin(2t) \]
   \[ y' - \frac{y}{t} = t^2 \sin(2t) \]
   has the form \[ y' + p(t)y = g(t) \]
   So linear and first order

2. \[ \frac{dy}{dx} - \frac{2y}{x} = x^{-1}y^{-1} \] \hspace{1cm} \text{This is a Bernoulli type} \hspace{1cm} (n = -1)
   Nonlinear and first order

3. \[ y'' + \cos(t + y) = e^t \]
   \[ 2^\text{nd} \text{ order due to } y'' \]
   Nonlinear due to \[ \cos(t + y) \]

(b) (6 points) So far we have studied \textit{Separable}, \textit{Bernoulli} and \textit{Exact} ODEs. Classify each of the following using these 3 categories. Provide a brief justification for your answer. If none apply, simply state \textit{NONE}.

1. \[ \frac{dy}{dx} = 2xy^2 + 4y \]
   \[ y' - 4y = 2x y^2 \]
   \[ \text{Bernoulli, } n = 2 \]

2. \[ (x^2 + y)dx + (x - \sin(y))dy = 0 \]
   \[ \frac{\partial}{\partial y} (x^2 + y) = 1 = \frac{\partial}{\partial x} (x - \sin(y)) = 1 \]
   So this one
   is \textit{exact}.

3. \[ \frac{dy}{dx} = e^{x+y} \]
   \[ \frac{dy}{dx} = e^x e^y \]
   So \textit{separable}. 


2. (a) (10 points) Solve

\[ \frac{dy}{dt} + 2y = \frac{t}{t^3 + 1} \]

This equation is linear.

\[ \mu(t) = e^{\int \frac{2}{t^3 + 1} \, dt} = e^{\ln(t^2 + 1)} = t^2. \quad \boxed{+2} \]

\[ (t^2 y)' = \frac{t^2}{t^3 + 1} \quad \boxed{+3} \]

\[ t^2 y = \int \frac{t^2}{t^3 + 1} \, dt \]

\[ t^2 y = \frac{1}{3} \ln |t^3 + 1| + C \quad \boxed{+5} \]

Let \( u = t^3 + 1 \)

\[ \frac{du}{dt} = 3t^2 \quad \Rightarrow \frac{1}{3} \, du = t^2 \, dt \]

So

\[ \frac{1}{3} \int \frac{1}{u} \, du = \frac{1}{3} \ln |u| + C \]

\[ = \frac{1}{3} \ln |t^3 + 1| + C \quad \boxed{+4} \]
(b) (10 points) Solve

\[
\frac{dy}{dt} = t(y^2 - 1) \\
y(0) = 1 \quad y(-1) = 0
\]

So this one is separable.

\[
\int \frac{dy}{y^2 - 1} = \int t \, dt
\]

LHS integral

\[
\int \frac{dy}{y^2 - 1} = \int \frac{1}{(y+1)(y-1)} \, dy
\]

Partial fraction decomposition.

\[
\frac{1}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}
\]

\[
\frac{1}{(y+1)(y-1)} = \frac{A(y-1) + B(y+1)}{(y+1)(y-1)}
\]

\[
1 = Ay-A + By+B,
\]

\[
1 = (A+B)y + (B-A)
\]

\[
B-A = 1 \quad \cdots (i)
\]

\[
A+B = 0 \quad \cdots (ii)
\]

\[
B = -A \quad \text{plug into (ii)}
\]

\[
-A-A = 1 \Rightarrow -2A = 1 \\
A = -\frac{1}{2}
\]

\[
B = \frac{1}{2}
\]

LHS integral become

\[
\frac{1}{2} \int \frac{1}{y-1} - \frac{1}{y+1} \, dy = \frac{1}{2} \left[ \ln |y-1| - \ln |y+1| \right]
\]

\(\Rightarrow\)

\[
\frac{1}{2} \ln \left| \frac{y-1}{y+1} \right| = \frac{t^2}{2} + c
\]

\[
y(0) = -1 \Rightarrow
\]

\[
\frac{1}{2} \ln \left| \frac{0-1}{0+1} \right| = \frac{1}{2} + c
\]

\[
\frac{1}{2} \ln 1 = \frac{1}{2} + c \quad c = -\frac{1}{2}
\]

\[
\frac{1}{2} \ln \left| \frac{y-1}{y+1} \right| = \frac{t^2}{2} + \frac{1}{2}
\]
(c) (10 points) Transform the equation
\[ \frac{dy}{dt} = -y + e^t y^2 \]
into a first order ODE. Do not solve the resulting ODE.

Let \( v = y^{2-1} = y^{-1} \)
\[ \frac{dv}{dt} = -y^{-2} \frac{dy}{dt} \]
\[ \Rightarrow -y^2 \frac{dv}{dt} = \frac{dy}{dt} \]
\[ -y^2 \frac{dv}{dt} + y = -e^t y^2 \]
\[ \frac{dv}{dt} - y^{-1} = -e^t \]
\[ \text{but } y^{-1} = v \text{ so } \]
\[ v' - v = -e^t \]
3. A 50 litre tank initially contains $q_0$ grams of salt. A salty mixture with a concentration $e^{-\frac{t}{25}}$ grams/litre at time $t$ flows in at a rate of 2 litres/minute, and a thoroughly stirred mixture comes out at the same rate.

(a) (15 points) Set up and solve a differential equation to determine the amount of salt at time $t$.

Let $Q(t)$ be the quantity of salt at time $t$.

$Q(0) = q_0$

\[
\frac{dQ}{dt} = \text{Rate in} - \text{Rate out}
\]

\[
= \left( e^{-\frac{t}{25}} \right) \left( 2 \text{litres/min} \right) - \left( \frac{Q(t)}{50} \right) \left( 2 \text{litres/min} \right)
\]

\[
= 2e^{-\frac{t}{25}} - \frac{Q}{25}
\]

\[
\frac{dQ}{dt} + \frac{Q}{25} = 2e^{-\frac{t}{25}}
\]

\[k(t) = e^{\int \frac{1}{25} dt} = e^{\frac{t}{25}}\]

\[
\frac{d}{dt} \left( Q e^{\frac{t}{25}} \right) = 2 e^{-\frac{t}{25}} e^{\frac{t}{25}}
\]

\[
\frac{d}{dt} \left( Q e^{\frac{t}{25}} \right) = 2, \quad \text{integrating}
\]

\[Q e^{\frac{t}{25}} = 2t + C, \quad Q(0) = q_0 \Rightarrow C = q_0
\]

\[Q e^{\frac{t}{25}} = 2t + q_0 \Rightarrow Q(t) = (2t + q_0) e^{-\frac{t}{25}}
\]

(b) (5 points) Suppose you discover a small leak in the tank and estimate it to be about 0.1 litres/minute. State but do not solve the new differential equation.

The leak changes the outflow

1. New rate of outflow = $2 + 0.1 = 2.1 \text{litres/min}$

2. Volume of tank decreases, $V(t) = 50 - 0.1t$

\[
\frac{dQ}{dt} = 2e^{-\frac{t}{25}} - \left( \frac{Q}{50 - 0.1t} \right) \left( 2.1 \text{litres/min} \right)
\]
4. Given
\[ \frac{dy}{dt} = -r(1 - \frac{y}{5})(1 - \frac{y}{10})y \]
where \( r > 0 \) is a constant.

(a) (13 points) Find and classify the equilibrium solutions to the ODE. Show your equilibrium solution on a phase line.

Equilibrium points:
\[-r\left(1 - \frac{y}{5}\right)(1 - \frac{y}{10})y = 0 \Rightarrow \left(\frac{-y}{5}\right) = 0 \text{ or } \left(\frac{y - 10}{10}\right) = 0, \text{ or } y = 0\]
\[y = 0, \quad y = 5, \quad y = 10\]

(b) (5 points) Suppose \( y(0) = 6 \). Determine \( \lim_{t \to \infty} y(t) \).

If \( y(0) = 6 \), \( \frac{dy}{dt} > 0 \) so \( y(t) \to 10 \).
5. (15 points) Solve \( (x^2 + y)dx + (x - \sin(y))dy = 0 \).

\[
M(x,y) = x^2 + y \quad \quad N(x,y) = x - \sin(y)
\]

\[
M_y = 1 \quad \quad N_x = 1 \quad \quad \text{so} \quad \text{Exact}
\]

Let \( \psi_x(x,y) = x^2 + y \)

\[
\psi(x,y) = \int (x^2 + y) \, dx = \frac{x^3}{3} + xy + h(y)
\]

\[
\frac{\partial}{\partial y} \left( \frac{x^3}{3} + xy + h(y) \right) = x + h'(y) = x - \sin(y).
\]

Therefore \( h'(y) = -\sin(y) \) \( \Rightarrow h(y) = \cos(y) \).

Our solution is

\[
\psi(x,y) = \frac{x^3}{3} + xy + \cos(y) = C.
\]
6. (10 points) Solve \( \frac{dy}{dx} = \frac{1}{e^y + x} \)

\[
(e^y + x) \, dy = dx \quad \Rightarrow \quad -dx + (e^y + x) \, dy = 0
\]

\( M = -1 \)

\( N = e^y + x \)

Is this Exact?

\( My = 0 \) and \( Nx = 1 \), so \( \text{NO} \).

Let's make it exact

\[
\mu(y) = e^{-y} \quad \Rightarrow \quad -\int_{-1}^{1} 0 \, dy
\]

\( \mu(y) = e^{-y} \quad \Rightarrow \quad -\int_{-1}^{1} 0 \, dy = 0 \)

\( -e^{-y} \, dx + e^{-y}(e^y + x) \, dy = 0 \)

\( \Rightarrow \) \( -e^{-y} \, dx + (1 + xe^{-y}) \, dy = 0 \)

\( k = e^{-y} \quad N = 1 + xe^{-y} \)

\( k' = e^{-y} \quad N_x = xe^{-y} \), \( k'y = N_x \), Yes this is exact.

Hence \( k(x, y) = -e^{-y} \) the \( \psi(x, y) = -e^{-y} + h(y) \)

\[
\frac{\partial}{\partial y} \left( -e^{-y} + h(y) \right) = e^{-y} x + h'(y) = 1 + xe^{-y}
\]

\( \Rightarrow \ h'(y) = 1 \) \( \Rightarrow \ h(y) = y \).

Our solution

\[
\psi(x, y) = -xe^{-y} + y = C
\]