1. (a) (8 points) Solve the initial value problem

$$y'' - y' - 2y = 0$$
$$y(0) = 1, y'(0) = \alpha$$

where α is a constant.

The characteristic polynomial is $r^2 - r - 2 = 0 \Leftrightarrow (r+1)(r-2) = 0$, $r = -1$, $r = 2$.

$y(t) = c_1 e^{-t} + c_2 e^{2t}$

$y(0) = 1 \Rightarrow c_1 + c_2 = 1 \cdots (i)$, $y'(t) = -c_1 e^{-t} + 2c_2 e^{2t}$

$y'(0) = \alpha \Rightarrow -c_1 + 2c_2 = \alpha. \cdots (ii)$

From (i) $c_2 - 1 = -c_1$, plug into (ii) $(c_2 - 1) + 2c_2 = \alpha$

$3c_2 = \alpha + 1$ $c_2 = \frac{1}{3}(\alpha+1)$

$c_1 = 1 - c_2 \Rightarrow c_1 = 1 - \frac{1}{3}\alpha - \frac{1}{3} = \frac{2}{3} - \frac{\alpha}{3}$

$y(t) = \left(\frac{2}{3} - \frac{\alpha}{3}\right) e^{-t} + \left(\frac{1}{3}(\alpha+1)\right) e^{2t}$

(b) (7 points) Find the value of α that ensures that $\lim_{t \to \infty} y(t) = 0$. Here $y(t)$ is the solution to the initial value problem above.

If $\lim_{t\to\infty} y(t) = 0$ A $\frac{1}{3}(\alpha+1) = 0 \Rightarrow \underline{\alpha = -1}$

Here the only way $y(t) \to 0$ is if $\frac{1}{3}(\alpha+1) = 0$ because

$e^{2t} \to \infty$ as $t \to \infty$.

2. (10 points) Find the form of the particular solution to

$$y'' + 2y' + 2y = e^{-t}\sin(t) + t^2e^{2t}$$

Do not solve for the coefficents - just write down what the particular solution should look like.

The complimentary solution

$y'' + 2y' + 2y = 0 \Rightarrow r^2 + 2r + 2 = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{2^2 - (4\cdot1\cdot2)}}{2\cdot1}$

$= \dfrac{-2 \pm \sqrt{-4}}{2}$

$= \dfrac{-2 \pm i\sqrt{4}}{2} = \dfrac{-2 \pm i2}{2}$

$-1 \pm i$

$$\boxed{y_c(t) = C_1e^{-t}\sin(t) + C_2e^{-t}\cos(t)}$$

Looking at the RHS, $e^{-t}\sin(t)$ is already a solution to the homogeneous problem so

$$y_p(t) = Ate^{-t}\sin(t) + Bte^{-t}\cos(t) + (C + Dt + Et^2)e^{2t}$$

3. (25 points) Find the general solution to

$$y'' - y = te^{2t}$$

$y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Leftrightarrow (r+1)(r-1) = 0, \Rightarrow r = \pm 1.$

$y_c(t) = c_1 e^{-t} + c_2 e^{t}$

Let $y_p(t) = (A_0 + A_1 t) e^{2t}$ because $r=2$ is not a root of char poly.

$y_p'(t) = (A_0 + A_1 t)(2e^{2t}) + A_1 e^{2t}$

$y_p''(t) = (A_0 + A_1 t)(4e^{2t}) + (2e^{2t}) A_1 + 2A_1 e^{2t}$

$y_p''(t) - y_p(t) = \underline{4A_0 e^{2t}} + \underline{4A_1 t e^{2t}} + 4A_1 e^{2t} - \underline{A_0 e^{2t}} - \underline{A_1 t e^{2t}}$

$= 3A_0 e^{2t} + 3A_1 t e^{2t} + 4A_1 e^{2t}$

$= (3A_0 + 4A_1) e^{2t} + 3A_1 t e^{2t} = te^{2t}$

Comparing coeffiaents

$3A_0 + 4A_1 = 0 \quad \cdots (i)$

$3A_1 = 1 \quad \cdots (ii)$

From (ii) $A_1 = \frac{1}{3}$, plug into (i) $\quad 3A_0 = -4A_1$

$= -4 \cdot \frac{1}{3} \Rightarrow A_0 = -\frac{4}{9}$

$y(t) = c_1 e^{t} + c_2 e^{-t} + \left(-\frac{4}{9} + \frac{1}{3} t\right) e^{2t}$

4. (25 points) Find the general solution to

$$y'' + 2y' + y = e^{-t}\ln(t)$$

$y''+2y'+y=0 \Rightarrow r^2+2r+1=0 \Leftrightarrow (r+1)^2=0$, $r=-1$, twice.

Therefore $y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$ so we have

$y_1(t) = e^{-t}$, $y_2(t) = te^{-t}$,

$$W[y_1,y_2](t) = (e^{-t})\{-te^{-t}+e^{-t}\} - (te^{-t})\{-e^{-t}\}$$

$$= \underline{-t(e^{-t})^2} + (e^{-t})^2 + \underline{t(e^{-t})^2} = (e^{-t})^2$$

Variation of parameters

$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

$$u_1(t) = -\int \frac{y_2(t)\,g(t)}{W[y_1,y_2](t)}\,dt \quad \text{and} \quad u_2(t) = \int \frac{y_1(t)\,g(t)}{W[y_1,y_2](t)}\,dt$$

$$u_1(t) = -\int \frac{(te^{-t})\,e^{-t}\ln(t)}{(e^{-t})^2}\,dt$$

$$= -\int t\ln(t)\,dt \underset{\text{by parts!}}{=} \frac{t^2}{2}\ln(t) - \int \frac{t^2}{2}\cdot\frac{1}{t}\,dt$$

Let

$u = \ln(t)$ $\quad dv = t\,dt$

$du = \frac{1}{t}dt$ $\quad v = \frac{t^2}{2}$

$$= \frac{t^2}{2}\ln(t) - \int \frac{t}{2}\,dt = \frac{t^2}{2}\ln(t) - \frac{t^2}{4}$$

$$u_2(t) = \int \frac{(e^{-t})(e^{-t}\ln(t))}{(e^{-t})^2}\,dt$$

$$= \int \ln(t)\,dt$$

Let $u = \ln(t)$ $\quad dv = dt$

$\frac{du}{dt} = \frac{1}{t}$ $\quad v = t$

$du = \frac{1}{t}dt$

$$\int \ln(t) = t\ln(t) - \int t\,\frac{1}{t}\,dt$$

$$= t\ln(t) - \int t\cdot\frac{1}{t}\,dt$$

$$= t\ln(t) - t$$

$$y(t) = c_1 e^{-t} + c_2 te^{-t} + \left[\left(\frac{t^2}{2}\ln(t) - \frac{t^2}{4}\right)e^{-t} + (t\ln(t) - t)\,te^{-t}\right]$$

5. (25 points) Ryan has solved an ode of the form

$$ay'' + by' + cy = f(t)$$

and found a general solution of the form

$$y(t) = c_1 e^{-2t} + c_2 e^{3t} + (3t + 1)$$

Find the ode that has been solved. Specifically, find the constants a, b and c and the function $f(t)$.

Recall that $y(t) = y_c(t) + y_p(t)$, here $y_c(t) = c_1 e^{-2t} + c_2 e^{3t}$ so we have $r_1 = -2$, $r_2 = 3$.

This means that our characteristic polynomial is $(r+2)(r-3) = 0$

$\Rightarrow$ $r^2 - r - 6 = 0$. Therefore the ode is

$$y'' - y' - 6y = f(t)$$

$y_p(t) = 3t+1$, ~~~~ We know that $f(t)$ is a linear function because $y_p(t)$ is linear and $r=0$ is not a root of the characteristic polynomial.

$y_p'(t) = 3$, $y_p''(t) = 0$, therefore

$$y_p'' - y_p' - 6y_p = 0 - (3) - 6(3t+1) = f(t) = At+B$$

$$-3 - 18t - 6 = f(t)$$

$$-9 - 18t = At + B \Rightarrow \begin{matrix} A = -18 \\ B = -9 \end{matrix}$$

This means that the ODE is

$$y'' - y' - 6y = -18t - 9.$$