INTRODUCTION

What is an ODE?

An equation containing some derivatives of an unknown function.

Examples

1. A falling object (Sky diver falling from a plane)

From Newton's second law

$$F = ma \quad \text{i.e} \quad \text{mass} \times \text{acceleration} = \text{sum of forces}$$

where m is the mass (kg) and a is the acceleration (m/s^2). Recall from calc I that

$$a = \frac{dv}{dt} \quad \text{so} \quad \boxed{F = m\frac{dv}{dt}} \qquad (1)$$

More details on F

(a) A free falling body is acted on by gravity $\rightarrow F_g = mg$ $\quad (g \cong 9.8\, m/s^2)$

(b) The object also experiences drag (we will assume that this force $\propto$ velocity)

so $F_d = -\gamma v$ where γ is the drag coefficient $\left(\frac{kg}{s}\right)$, $v\,(m/s)$

The total force

$$\boxed{F = F_g + F_d = mg - \gamma v} \qquad (2)$$

Combining (1) and (2) yeilds an ODE

$$\boxed{m\frac{dv}{dt} = mg - \gamma v} \qquad (3)$$

Remarks:

1. The unknown function is velocity, v
2. m, g and γ are constant $\rightarrow$ generally reffered to as PARAMETERS.
3. Our objective is to find $v(t)$ satisfying (3)

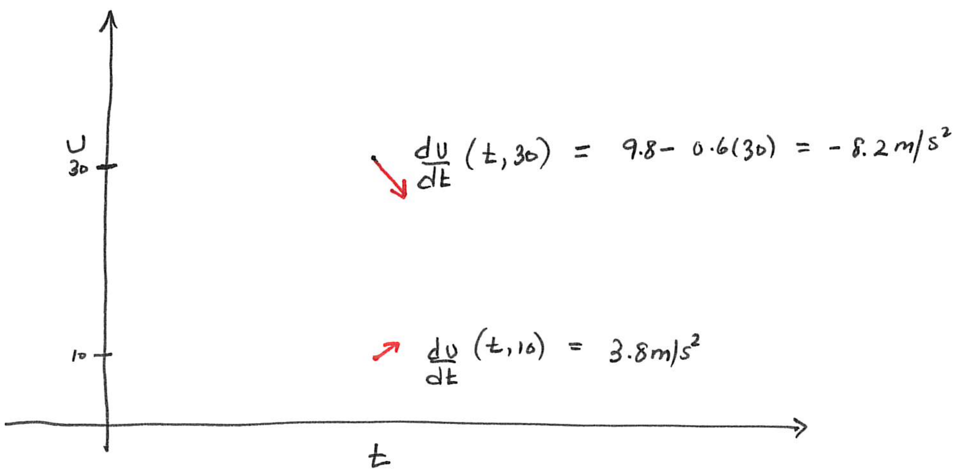
From (3)

$$\frac{dv}{dt} = g - \frac{\gamma v}{m} = \underbrace{f(t,v)}_{\text{Rate function}^{*}}$$

* Evaluation of $f(t,v)$ yields slope information for $v(t)$ [the solution we seek]

eg: setting $\boxed{\begin{aligned} m &= 72kg \\ g &= 9.8 m/s^2 \\ \gamma &= 43.2 kg/s \end{aligned}}$ so that $\boxed{\frac{dv}{dt} = 9.8 - 0.6v}$

(t, v) plane



Repeating evaluation for more points in (t v) yields a <u>SLOPE field</u> / <u>Direction field.</u>

<u>Matlab attachment</u> (dirfield.m)

1. Slope field shows accelaration on t-y plane
2. Each line on the slope field is tangent to $v(t)$
3. Observe that if $v <$ some v^*, $\frac{dv}{dt} > 0$ and if $v > v^*$, $\frac{dv}{dt} < 0$

4. v^* separates divers that are accelarating from those decelarating

<u>Key Question</u>

What value of v yields $\frac{dv}{dt} = 0$? set $mg - \gamma v = 0 \Rightarrow v = \frac{mg}{\gamma}$

In our case:

$$\frac{dv}{dt} = 9.8 - 0.6v \underset{\substack{\uparrow \\ \frac{dv}{dt} = 0}}{\Rightarrow} 9.8 - 0.6v = 0 \Rightarrow \frac{9.8}{0.6} = v = 16\tfrac{1}{3} m/s$$

$v(t) = 16\frac{1}{3} m/s$ is called the <u>equilibrium solution</u> (represents balance between gravity & drag).

5. All solutions converge to the equilibrium solution (as $t \to \infty$)

$\Rightarrow$ $v(t) = \frac{mg}{\gamma}$ is called the terminal velocity. $\left[\frac{dv}{dt} = g - \frac{\gamma v}{m}, \quad \frac{dv}{dt} = 0 \Rightarrow mg = \gamma v, \quad \frac{mg}{\gamma} = v \right]$

2 Population of mice

$$\boxed{\frac{dp}{dt} = rp - k}$$

r - growth rate (mice/month)

k - predation term (assuming the presence of owls).

Matlab attachment ($k = 450$, $r = 0.5$)

1. Direction field in tp plane show equilibrium at

$$\frac{dp}{dt} = 0 \Rightarrow rp - k = 0 \quad \boxed{p = \frac{r}{k}}$$

2. For large p, $\frac{dp}{dt} > 0$ so p increases

for small p, $\frac{dp}{dt} < 0$ so p decreases.

3. Solutions diverge from the equilibrium solution.

3 Neurology (Slides)

$$x'(t) = -x + s(x - \theta + e(t))$$

$x(t)$ - percentage of nerve cells active in the brain.

$e(t)$ - activity from cells outside the region

θ - threshold level of cells in the region $\quad e(t) = -0.3\cos(2\pi t)$

S - response function $\quad S(z) = \frac{1}{1+e^{-kz}} \quad k = 15$

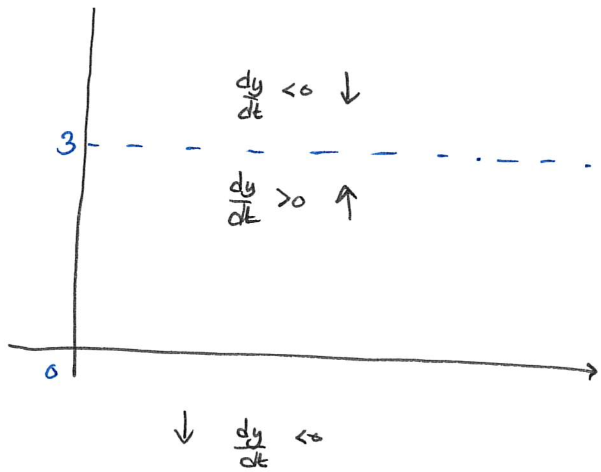
$$x'(t) = -x(t) + \frac{1}{1+e^{-15(x(t) - 0.5 - 0.3\cos(2\pi t))}}$$

Matlab attachment

1. Oscillatory solution near 50%
2. If x is large $x(t)$ is near 100%
3. If x is small $x(t)$ is near 0.

Other direction fields

(a) $y' = y(3-y)$



$\frac{dy}{dt} = 0 \Rightarrow y(3-y) = 0$

$y = 0$, $y = 3$ are equillibrium solutions

- Solutions converge to $y=3$ for $y(0) > 0$
- Solutions diverge to $-\infty$ for $y(0) < 0$

(b) $y' = e^{-t} + y$

If $y(0) > 0$, solutions approach $+\infty$

If $y(0) < 0$, solutions approach $-\infty$

If $y(0) = 0$, solutions approach 0.