What is an ODE?

An equation containing some derivatives of an unknown function.

Examples

1. A falling object (Sky diver falling from a plane)

From Newton's second law

\[ F = ma \quad i.e. \quad \text{mass} \times \text{acceleration} = \text{sum of forces} \]

where \( m \) is the mass (kg) and \( a \) is the acceleration (m/s^2). Recall from calc I that

\[ a = \frac{dv}{dt} \quad \text{so} \quad F = m \frac{dv}{dt} \quad (1) \]

More details on \( F \)

(a) A free falling body is acted on by gravity \( F_g = mg \) (\( g \approx 9.8 \text{m/s}^2 \))

(b) The object also experiences drag \( F_d \) (we will assume that this force \( \propto \) velocity)

so

\[ F_d = -\gamma v \quad \text{where} \quad \gamma \text{ is the drag coefficient (kg/s)} \quad , \quad v \text{ (m/s)} \]

The total force

\[ F = F_g + F_d = mg - \gamma v \quad (2) \]

Combining (1) and (2) yields an ODE

\[ m \frac{dv}{dt} = mg - \gamma v \quad (3) \]

Remarks:

1. The unknown function is velocity \( v \)
2. \( m, g \) and \( \gamma \) are constant \( \Rightarrow \) generally referred to as parameters.
3. Our objective is to find \( v(t) \) satisfying (3).
\[
\frac{du}{dt} = g - \frac{\gamma v}{m} = f(t, u)
\]

Rate function

Evaluation of \( f(t, u) \) yields slope information for \( u(t) \) [the solution we seek]

Example setting:
- \( m = 72 \text{ kg} \)
- \( g = 9.8 \text{ m/s}^2 \)
- \( \gamma = 43.2 \text{ kg/s} \)

So that
\[
\frac{du}{dt} = 9.8 - 0.6v
\]

\((t, u)\) plane

\[
\frac{du}{dt} (t, 30) = 9.8 - 0.6(30) = -0.2 \text{ m/s}^2
\]

\[
\frac{du}{dt} (t, 10) = 3.8 \text{ m/s}^2
\]

Repeating evaluation for more points in \((t, u)\) yields a slope field / direction field.

Matlab attachment (dirfield.m)

1. Slope field shows acceleration on \( t \)-\( y \) plane
2. Each line on the slope field is tangent to \( u(t) \)
3. Observe that if \( v < u^* \), \( \frac{du}{dt} > 0 \) and if \( v > u^* \), \( \frac{du}{dt} < 0 \)
4. \( u^* \) separates divers that are accelerating from those decelerating

Key Question

What value of \( u \) yields \( \frac{du}{dt} = 0 \)?

Set \( mg - \gamma u = 0 \) \( \Rightarrow \) \( u = \frac{mg}{\gamma} \)

In our case,
\[
\frac{du}{dt} = 9.8 - 0.6v \Rightarrow 9.8 - 0.6v = 0 \Rightarrow \frac{9.8}{0.6} = u = 16\frac{1}{3} \text{ m/s}
\]

\( u(t) = 16\frac{1}{3} \text{ m/s} \) is called the equilibrium solution (represents balance between gravity & drag).
5. All solutions converge to the equilibrium solution \((\alpha t \to \infty)\)

\[ U(t) = \frac{mg}{\gamma} \]

is called the terminal velocity.

\[
\begin{align*}
\frac{dv}{dt} &= mg - \gamma \frac{v}{m}, \\
\frac{dv}{dt} &= 0 \\
\frac{mg}{\gamma} &= v
\end{align*}
\]

\[ \text{Population of mice} \]

\[
\frac{dp}{dt} = rp - k
\]

- \( r \) - growth rate (mice/month)
- \( k \) - predation term (assuming the presence of owls).

\[ \text{Matlab attachment:} \quad \left( k = 450, \quad r = 0.5 \right) \]

1. Direction field in \( tp \) plane show equilibrium at

\[
\frac{dp}{dt} = 0 \Rightarrow rp - k = 0 \quad \boxed{p = \frac{r}{k}}
\]

2. For large \( p \), \( \frac{dp}{dt} > 0 \) so \( p \) increases.

3. For small \( p \), \( \frac{dp}{dt} < 0 \) so \( p \) decreases.

3. Solutions diverge from the equilibrium solution.

\[ \text{Neurology (Slides)} \]

\[ x'(t) = -x + S(x - \theta + e(t)) \]

- \( x(t) \) - percentage of neurons active in the brain.
- \( e(t) \) - activity from cells outside the region.
- \( \theta \) - threshold level of cells in the region.
- \( e(t) = -0.3 \cos(2\pi t) \)
- \( S \) - response function

\[ S(z) = \frac{1}{1 + e^{-kz}} \quad k = 15 \]

\[ x'(t) = -x(t) + \frac{1}{1 + e^{-15(x(t) - 0.5 - 0.3 \cos(2\pi t))}} \]

Matlab attachment

1. Oscillatory solution near 50%
2. If \( x \) is large \( x(t) \) is near 100%
3. If \( x \) is small \( x(t) \) is near 0.
Other direction fields

(a) \( y' = y(3-y) \)

- \( \frac{dy}{dt} = 0 \Rightarrow y(3-y) = 0 \)
  - \( y = 0 \), \( y = 3 \) are equilibrium solutions

- Solutions converge to \( y = 3 \) for \( y(0) > 0 \)
- Solutions diverge to \( -\infty \) for \( y(0) < 0 \)

(b) \( y' = e^{-t} + y \)

- If \( y(0) > 0 \), solutions approach \( +\infty \)
- If \( y(0) < 0 \), solutions approach \( -\infty \)
- If \( y(0) = 0 \), solutions approach 0.