1.2 **Some analytic solutions**

So far we have mostly ODEs of the form

$$\boxed{\frac{dy}{dt} = ay - b}$$

where a and b are constants.

Check

① $\frac{dv}{dt} = g - \frac{\gamma}{m} v$ (falling sky-diver)

② $\frac{dp}{dt} = rp - k$ (Mice population)

Analytic solution for sky diver model

$$\frac{dv}{dt} = 9.8 - 0.6v$$

$$= -0.6\left(-16\tfrac{1}{3} + v\right)$$

$$\frac{\frac{dv}{dt}}{v - 16\frac{1}{3}} = -0.6$$

* key! find a function s.t the derivative is $\frac{1}{v-16\frac{1}{3}} \cdot \frac{dv}{dt}$.

Chain Rule!

Recall that $\frac{d}{dt} \ln|v| = \frac{1}{v} \cdot \frac{dv}{dt}$

$$\frac{d}{dt}\left(\ln\left|v - 16\tfrac{1}{3}\right|\right) = -0.6$$

Integrating both sides w.r.t t yields

$$\ln\left|v - 16\tfrac{1}{3}\right| = -0.6t + C$$

$$\left|v - 16\tfrac{1}{3}\right| = e^{-0.6t + C} = e^{C} \cdot e^{-0.6t} \Leftrightarrow v - 16\tfrac{1}{3} = \pm e^{C} e^{-0.6t}$$

(*) $\boxed{v(t) = 16\tfrac{1}{3} + ce^{-0.6t}}$ * $c = \pm e^{C}$ (This is the general solution).

1. Notice that $v(t) = 16\frac{1}{3}$ m/s (the equilibrium solution) is a solution for $c = 0$.
2. (*) represents an infinite number of solutions for each value of c
3. (*) can be represented geometrically by integral curves

To obtain a specific solution we need an initial condition

eg $v(0) = 0$ i.e the sky diver drops out of the plane.

General Solution
$$v(t) = 16\tfrac{1}{3} - Ce^{-0.6t}$$
$$v(0) = 0 \Rightarrow 0 = 16\tfrac{1}{3} - Ce^{-0.6(0)} \Rightarrow C = 16\tfrac{1}{3}$$

$$v(t) = 16\tfrac{1}{3}\left(1 - e^{-0.6t}\right)$$

* Notice that $\lim_{t \to \infty} v(t) = 16\tfrac{1}{3}(1-0) = 16\tfrac{1}{3}$!

Suppose the sky-diver drops from a height of 4000m. ~~[illegible]~~
Find their velocity after falling 1000m.

We have $v(t) = 16\tfrac{1}{3}\left(1 - e^{-0.6t}\right)$ so we need to find how long it takes the diver to fall 1000m.

Let x be the distance the diver has fallen, then

$$\frac{dx}{dt} = v = 16\tfrac{1}{3}\left(1 - e^{-0.6t}\right) = 16\tfrac{1}{3} - 16\tfrac{1}{3}e^{-0.6t}$$

Integrating both sides w.r.t t

$$x(t) = 16\tfrac{1}{3}t + \frac{245}{9}e^{-0.6t} + k$$

Assuming $x(0) = 0$ $\quad k = -\frac{245}{9}$ so that

$$\boxed{x(t) = \frac{49}{3}t + \frac{245}{9}e^{-0.6t} - \frac{245}{9}}$$

T s.t $x(T) = 1000m$

$$1000m = \frac{49}{3}t + \frac{245}{9}e^{-0.6T} - \frac{245}{9}$$ * Matlab! fzero

$\underline{T = 7.78s}$

$$v(T) = 16\tfrac{1}{3}\left(1 - e^{-0.6\cdot(7.78)}\right) = 16.17m/s.$$

1.2 Mouse population

Suppose $r = 0.5$ and $k = 450$ so that $\frac{dp}{dt} = rp - k$ is

$$\frac{dp}{dt} = 0.5p - 450 = \frac{1}{2}(p - 900) \quad \text{and} \quad p(0) = 850$$

Solving

$$\frac{dp}{p-900} = \frac{1}{2}$$

$$\int \frac{dp}{p-900} = \frac{1}{2}\int dt$$

$$\ln(|p-900|) = \frac{t}{2} + c.$$

$$|p-900| = e^{\frac{t}{2}+c} \Rightarrow p - 900 = \pm e^{c} \cdot e^{\frac{t}{2}}$$

$$\boxed{p = 900 + Ce^{t/2}} \quad (*) \qquad p = 900 + Ce^{t/2}, \quad C = \pm e^{c}.$$

(*) is called the general solution of the initial value problem.

Geometrically (*) corresponds to an infinite number of solutions ~~for~~ each corresponding to a value of c.

Using the initial condition, we can isolate C.

$$p(0) = 850 \Rightarrow$$

$$850 = 900 + Ce^{0/2} \Rightarrow C = -50$$

$$p(t) = 900 - 50e^{t/2}$$

Matlab attachment

1. Notice that the equilibrium solution is @ $\frac{dp}{dt} = 0 \Rightarrow 0.5p - 450 = 0$
$p = 900$.

2. On either side of the equilibrium solution observe divergence ~~to~~ ~~[illegible]~~ of solutions away from $p = 900$ depending on the value of c.