Method of integrating factors

Solution technique for solving $\frac{dy}{dt} + p(t)y = g(t)$ (linear first order equations)

i.e. linear in $y(t)$ and first order because the highest derivative is y'.

Examples ① and ② are linear first order equations.

① $y' = -3y + \sin(t)$ can be written in the form

$$y' + 3y = \sin(t)$$

$p(t) = 3$, $g(t) = \sin(t)$

② $y' = y\sin(t) + t^3$ can be written in the form

$$y' - y\sin(t) = t^3$$

$p(t) = \sin(t)$, $g(t) = t^3$

③ $4y' + e^y t = 0$ is NOT linear in y because e^y is not linear in y.

④ $y y' = 2t$ is not linear because to write it in the standard form yeilds

$$y' = \frac{2}{y}t \quad \text{non-linear in } y.$$

Solving $\boxed{y' + p(t)y = g(t)}$

Objective: Write the left hand side as $\frac{d}{dt}(\ldots) = g(t)$ and integrate on both sides w.r.t t.

In some cases this is easy to do:

e.g

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t$$

Notice that $\frac{d}{dt}\left((4+t^2)y\right) \underset{\text{product Rule}}{=} (4+t^2)\frac{dy}{dt} + y \cdot 2t$

Then

$$\frac{d}{dt}\left((4+t^2)\,y\right) = 4t \quad , \quad \text{Integrating both sides}$$

$$(4+t^2)y = \frac{4t^2}{2} + c \quad \Rightarrow \quad y = \frac{2t^2+c}{(4+t^2)}$$

In general, given

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

multiply on both sides by $\mu(t)$ so that we can write the LHS as a derivative of a product :

$$\mu(t)\frac{dy}{dt} + \mu(t)\,p(t)\,y = \mu(t)\,g(t)$$

Observe that

If $\boxed{\mu(t)\,p(t) = \mu'(t)}$ $*$ then the LHS can be written as

$$\frac{d}{dt}\left[\mu(t)y(t)\right]$$

Check :

$$\frac{d}{dt}\left[\mu(t)y(t)\right] = \mu(t)\frac{dy}{dt} + y(t)\frac{d\mu(t)}{dt} = \mu(t)\frac{dy}{dt} + y(t)\,\mu(t)\,p(t).$$

$$\mu'(t) = \mu(t)p(t)$$

To find $\mu(t)$ we solve the ODE $(*)$

$$\frac{d\mu(t)}{dt} = \mu p$$

$$\frac{\frac{d\mu}{dt}}{\mu} = p \quad \Rightarrow \quad \frac{d}{dt}\ln(\mu) = p(t)$$

$$\ln(\mu) = \int p(t)\,dt$$

$$\mu = e^{\int p(t)\,dt}.$$

$\mu(t) = e^{\int p(t)\,dt}$ is called the integrating factor.

Example #1. Solve

$$\frac{dy}{dt} + 2y = te^{-2t}$$

(Note the form $\frac{dy}{dt} + p(t)y = g(t)$)

Solution

Multiply by $\mu(t) = e^{\int p(t)\,dt} = e^{\int 2\,dt} = e^{2t}$ on both sides:

$$\left(\frac{dy}{dt} + 2y\right)e^{2t} = te^{-2t}\cdot e^{2t}$$

$$\frac{d}{dt}\left(e^{2t}\cdot y\right) = t$$, integrating both sides w.r.t t

$$e^{2t}\cdot y = \int t\,dt = \frac{t^2}{2} + c$$

$$y(t) = e^{-2t}\left(\frac{t^2}{2} + c\right)$$ ← this is the general solution. If we have an initial condition, say $y(0)=1$ ⇒ $t=0, y=1$ ⇒ $1 = e^{-2\cdot 0}\left(\frac{0^2}{2} + c\right)$ ⇒ $\underline{c = 1.}$

Example #2

Solve

$$\frac{dy}{dx} + y = \sqrt{1+\cos^2 x}, \quad y(1) = 4.$$

and find the value of ~~y(2)~~ $y(2)$.

(Note the form $\frac{dy}{dx} + p(x)y = g(x)$ here x has replaced t)

Solution

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int 1\,dx} = e^{x}.$$

$$\left(\frac{dy}{dx} + y\right)e^{x} = e^{x}\left(\sqrt{1+\cos^2 x}\right)$$

$$\frac{d}{dx}\left(e^{x}\cdot y\right) = e^{x}\left(\sqrt{1+\cos^2 x}\right)$$, integrating on both sides w.r.t x from $x=1$ to 2

$$e^{x}y\Big|_{x=1}^{2} = \underbrace{\int_1^2 e^{x}\left(\sqrt{1+\cos^2 x}\right)dx}$$

If we had no limits, the general solution is

$$y(x) = e^{-x}\left(\int e^{x}\sqrt{1+\cos^2 x}\,dx + C\right).$$

the indefinite integral does not an expression involving elementary functions so we approximate using MATLAB. >> "integral". function in MATLAB.

$$e^2 y(2) - e^1 y(1) \cong 4.8414$$

$$e^2 y(2) \cong e^1 y(1) + 4.8414 \Rightarrow y(2) \cong e^{-2+1}\cdot 4 + e^{-2}\cdot 4.8414 \cong 2.127$$

Example

Find the general solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos(x)$$

$x > 0$.

$$\left(\begin{array}{l} \text{General form} \\ \frac{dy}{dx} + p(x)y = g(x) \end{array} \right)$$

Multiply by x to convert into the general form.

$$(*) \quad \frac{dy}{dx} - \frac{2}{x}y = x^2\cos(x) \longleftarrow \boxed{p(x) = -\frac{2}{x}}$$

Integrating factor is $e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = e^{\ln(x^{-2})} = x^{-2}$.

Multiply $(*)$ by x^{-2}

$$\underbrace{x^{-2}\frac{dy}{dx} - 2x^{-3}y}_{\frac{d}{dx}\left(x^{-2}y\right)} = \cos(x)$$

$$\frac{d}{dx}\left(x^{-2}y\right) = \cos(x)$$

Integrating both sides w.r.t x

$$x^{-2}y = \int \cos(x)\,dx = \sin(x) + C$$

$$y = x^2\sin(x) + Cx^2$$

Show that $y = \dfrac{2t^2+c}{4+t^2}$ is a solution to $(4+t^2)y' + 2ty = 4t$

First, compute y' as

$$y' = \frac{(4+t^2)(4t) - (2t^2+c)(2t)}{(4+t^2)^2} \quad \text{via Quotient Rule.}$$

$$= \frac{(16t + 4t^3) - (4t^3 + 2ct)}{(4+t^2)^2} = \frac{16t - 2ct}{(4+t^2)^2}$$

Now plug into LHS of ODE and do a bit of algebra

$$(4+t^2)\left(\frac{16t - 2ct}{(4+t^2)^2}\right) + 2t\left(\frac{2t^2+c}{4+t^2}\right)$$

$$= \frac{16t - 2\cancel{c}t}{(4+t^2)} + \frac{4t^3 + 2\cancel{c}t}{4+t^2}$$

Notice here that the terms with c cancel so in theory we can pick c=0 to make the algebra easier!

$$= \frac{16t + 4t^3}{(4+t^2)} = \frac{4t\cancel{(4+t^2)}}{\cancel{(4+t^2)}}$$

$$= 4t. \checkmark$$

← This is the RHS of the ode!