2.2

Separable Differential Equations

We consider

$$\frac{dy}{dx} = f(x,y)$$

* here x has replaced t in comparison to 2.1

If the rhs function $f(x,y)$ can be expressed as $g(x)p(y)$ then the ODE is separable.

Examples

① $\frac{dy}{dx} = \frac{2x+xy}{y^2+1}$ is separable

Notice that $\frac{2x+xy}{y^2+1} = x \cdot \left(\frac{2+y}{y^2+1}\right) = g(x)p(y)$.

② $\frac{dy}{dx} = 1+xy$ is NOT separable.

SOLVING SEPARABLE EQUATIONS

Example

1. Solve $\frac{dy}{dx} = \frac{y-1}{x+3}$ $\quad y(-1)=0$

① Divide by $(y-1)$ and "multiply by dx"

$$\frac{dy}{y-1} = \frac{dx}{x+3}$$

* Notice that we have separated terms involving x and y on either side of the equality.

② Integrate on both sides

$$\int \frac{dy}{y-1} = \int \frac{dx}{x+3} \Rightarrow \ln|y-1| = \ln|x+3| + C$$

Applying the initial condition to determine C
i.e. ($y=0$ and $x=-1$)!

$y(-1)=0 \Rightarrow \ln|1| = \ln|2| + C \Rightarrow C = 0 - \ln 2 \quad C = -\ln(2)$

so $\ln(1-y) = \ln(x+3) - \ln(2)$

* Notice that for y near 0, $y-1<0$ so we replace ~~ln|y|~~ $(y-1)$ by the positive $(1-y)$

$$\ln(1-y) = \ln\left(\frac{x+3}{2}\right) \iff 1-y = \frac{x+3}{2}$$

The solution is $\boxed{y = 1 - \frac{1}{2}(x+3)}$

2. Solve $y' = \dfrac{ty(4-y)}{1+t}$, $y(0) = y_0 > 0$ and determine the behaviour of the solution as $t \to \infty$

First, notice that the rhs is separable

$$\frac{dy}{dt} = \frac{ty(4-y)}{1+t} = \left(\frac{t}{1+t}\right)\left(y(4-y)\right)$$

$$\frac{dy}{y(4-y)} = \frac{t}{1+t}\,dt$$ and integrating both sides

$$\int \frac{1}{y(4-y)}\,dy = \int \frac{t}{1+t}\,dt$$

$$\frac{t}{1+t} = \frac{1+t-1}{1+t} = 1 - \frac{1}{1+t}$$

$$\int \frac{t}{1+t}\,dt = \int 1 - \frac{1}{1+t}\,dt$$

$$= t - \ln|1+t| + C$$

Partial fractions

$$\frac{1}{y(4-y)} = \frac{A}{y} + \frac{B}{4-y}$$

$$\frac{1}{y(4-y)} = \frac{A(4-y) + B(y)}{y(4-y)}$$

$$1 = 4A - Ay + By$$
$$= 4A + (B-A)y$$

Comparing coefficients

$4A = 1 \Rightarrow A = \frac{1}{4}$

$B - A = 0 \Rightarrow B = A \Rightarrow B = \frac{1}{4}$.

$$\int \frac{1}{y(4-y)} = \int \frac{1}{4y} + \frac{1}{4(4-y)}\,dy$$

$$= \frac{1}{4}\ln|y| - \frac{1}{4}\ln|4-y|$$

combining LHS and RHS

$$\frac{1}{4}\ln\left|\frac{y}{4-y}\right| = t - \ln|1+t| + c$$

$$\ln\left|\frac{y}{4-y}\right| = 4t - 4\ln|1+t| + c$$

$$\left|\frac{y}{4-y}\right| = e^{(4t - 4\ln|1+t| + c} = \frac{e^{c}\cdot e^{4t}}{e^{\ln|1+t|^4}} = \frac{ce^{4t}}{(1+t)^4}$$

As $t\to\infty$, $\frac{e^{4t}}{(1+t)^4} \to \infty$ L'hopital's Rule.

so $\left|\frac{y}{4-y}\right| = \left|\frac{y}{y-4}\right| = \left|\frac{y-4+4}{y-4}\right| = \left|1+\frac{4}{y-4}\right| \to \infty$

we need $y(t) \to 4$.

Example

Solve $y' = \dfrac{2x}{1+2y}$, $y(2)=0$.

$f(x,y) = \dfrac{2x}{1+2y}$ is separable

$$\frac{dy}{dx} = \frac{2x}{1+2y}$$

$$(1+2y)\,dy = 2x\,dx$$

$y + \dfrac{2y^2}{2} = \dfrac{2x^2}{2} + c$. $y(2) = 0 \Rightarrow 0 = 2^2 + c$ $c = -4$

so

$$y + y^2 = x^2 - 4.$$

To obtain an explicit solution in y, solve the quadratic equation

$$y^2 + y - (x^2-4) = 0$$

Using the quadratic formula :

$$y = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-x^2+4)}}{2\cdot 1}$$

$$= \frac{-1 \pm \sqrt{4x^2 - 16}}{2} = -\frac{1}{2} \pm \sqrt{x^2 - \frac{15}{4}}$$

We have 2 solutions $\Rightarrow$ $y = -\frac{1}{2} \pm \sqrt{x^2 - \frac{15}{4}}$ pick the solution that satisfies the initial condition $y(2) = 0$

$$y = -\frac{1}{2} + \sqrt{x^2 - \frac{15}{4}}.$$

The other solution, $y = -\frac{1}{2} - \sqrt{x^2 - \frac{15}{4}}$ satisfies $y(2) = -1$

The solution is valid for $\left(x^2 - \frac{15}{4}\right) > 0 \Rightarrow x^2 > \frac{15}{2}$

$$|x| > \sqrt{\frac{15}{2}}$$

See Matlab attachment from dirfield.m.