2.3 Modeling with first-order ODEs

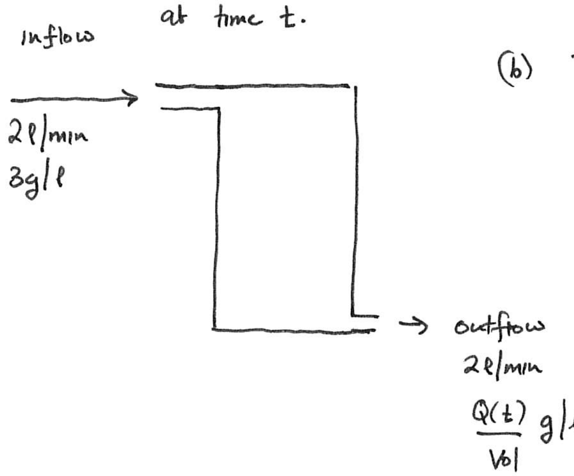
1 Mixing Problems

A fish tank initially contains 150ℓ of water with 20g of salt dissolved in it. The concentration in the tank needs to be increased from $\frac{20}{150}$ g/ℓ to 1g/ℓ to accomodate a new fish. Water containing 3g of salt per litre is allowed to flow into the tank at a rate of 2ℓ per minute. The thoroughly stirred mixture is allowed to flow out at the same rate.

1. Set up an initial value problem that describes the amount of salt $Q(t)$ in the tank at time t.

2. Use the initial condition $Q(0) = 20$ to find the time it will take to increase the salt concentration to 1g per litre.

SOLUTION : (a) Let $Q(t)$ denote the number of grams of salt in the tank at time t.



(b) The concentration of salt at any time, t is $\frac{Q(t)}{V}$ g/ℓ. (Assuming the mixture is thoroughly mixed)

$$\frac{dQ}{dt} = \text{Rate in} - \text{Rate out}$$

$$= (2\ell/\text{min})(3g/\ell) - (2\ell/\text{min}) \cdot \left(\frac{Q(t)}{\text{vol}}\ g/\ell\right)$$

$$\frac{dQ}{dt} = 6 - 2 \cdot \frac{Q(t)}{150} = 6 - \frac{Q(t)}{75}$$

ODE: $\boxed{\frac{dQ}{dt} = 6 - \frac{Q(t)}{75} \quad Q(0) = 20}$

Note

The ODE is linear and seperable so we can use either integrating factor technique or seperable technique

2.3

$$\frac{dQ}{dt} = \frac{450-Q}{75}$$

$$\frac{dQ}{450-Q} = \frac{1}{75}\,dt$$

$$\int \frac{dQ}{450-Q} = \frac{1}{75}\int dt$$

$$-\ln|(450-Q)| = \frac{t}{75}+c \iff \ln\left|\left(\frac{1}{450-Q}\right)\right| = \frac{t}{75}+c$$

Exponentiating on both sides

$$\frac{1}{450-Q} = \pm e^{\frac{t}{75}+c} = \pm e^{c}\cdot e^{\frac{t}{75}} = Ke^{\frac{t}{75}}, \quad K = \pm e^{c}$$

Solving for Q

$$\frac{1}{450-Q} = Ke^{\frac{t}{75}} \Rightarrow 450-Q = \frac{1}{K}e^{-\frac{t}{75}}$$

$Q(0) = 20 \Rightarrow$

$20 = 450 - \frac{1}{K}e^{0}$

$\frac{1}{K} = 450-20 = 430$

$K = \frac{1}{430}$

$$\boxed{Q(t) = 450 - \frac{430}{\cancel{1/K}}e^{-\frac{t}{75}}}$$

Note

$$\lim_{t\to\infty}(Q(t)) = \lim_{t\to\infty}\left(450 - \frac{1}{K}e^{-\frac{t}{75}}\right) = 450g$$

$$\text{concentration} = \frac{450}{150} = 3g/\ell.$$

To find the time it will take for the concentration to reach 1g/litre

We need the concentration, $\frac{Q(t)}{150} = 1$

$$\frac{1}{150}\left(450 - \frac{430}{\cancel{1/K}}e^{-\frac{t}{75}}\right) = 1$$

>> f = @(t) $\left(\frac{1}{150}\left(450 - 430e^{-\frac{t}{75}}\right) - 1\right)$

>> fzero(f, 20)

Initial guess.

Solving (MATLAB) yeilds $t \cong 27$ mins

(1b) They do

Suppose the rate of inflow is increased to 2.5 ℓ/min but the outflow cannot be increased.

The tank holds a maximum of 160 litres. Will the salt concentration reach the desired level before the container overflows?

Applications – Newton's Law of cooling

#1 Pepsi can

A cold Pepsi is taken out of a 40° fridge and placed on a picnic table. Five minutes later the Pepsi can has warmed up to 50°. If the outside temperature remains at 90°, what happens to the temperature of the can after 20 minutes? What about as $t \to \infty$?

Newton's law of cooling

If a small body of temperature T is placed in a room with constant temperature A

$$\frac{dT}{dt} \propto A - T \qquad \text{so that}$$

$$\frac{dT}{dt} = k(A-T).$$

k is a physical constant that depends on the physical properties of the object.

In our case:

$$\frac{dT}{dt} = k(90-T), \quad T(0) = 40°, \quad T(5) = 50$$

Our equation is separable, therefore

$$\int \frac{dT}{90-T} = \int k\,dt$$

$$-\ln|90-T| = kt + c$$

$$\ln|90-T| = -(kt+c)$$

$$|90-T| = e^{-(kt+c)} \Rightarrow 90-T = Ce^{-kt}, \quad C = \pm e^{-c}$$

Solving for C and k using the data points $T(0)=40$ and $T(5)=50$ yeilds:

$$T(0)=40 \Rightarrow 90-40 = Ce^{-k(0)} \Rightarrow \underline{C=50}$$

$$T(5)=50 \Rightarrow 90-50 = 50e^{-kt}$$

$$\frac{4}{5} = e^{-5k} \qquad k = \frac{\ln\left(\frac{4}{5}\right)}{-5}, \quad k = 0.04463$$

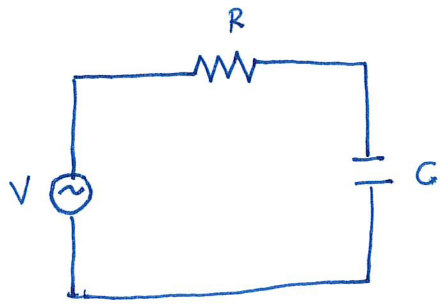
so $\quad 90-T = 50e^{-0.04463t} \Rightarrow T = 90-50e^{-0.04463t}$

$$T(20) = 90 - 50e^{-0.04463(20)} = 69.5^\circ.$$

As $t \to \infty$,

$$\lim_{t \to \infty} \left(90 - 50e^{-0.04463t} \right) = 90^\circ.$$

RC Circuit



R - resistor measured in ohms (Ω) - controls the flow of current

C - stores electric charge. measured in farads (F).

Kirchhoff's Laws

(A) Current

The algebraic sum of current flowing into any junction point must be zero.

(B) VOLTAGE

The algebraic sum of instantaneous changes in voltage drops around any closed loop must be zero.

Starting with the voltage law

$$U_R + U_C = U(t) \qquad (*)$$

From Ohm's Law

$$U_R = RI \qquad (\text{voltage drop} \propto \text{current passing through})$$

$\boxed{R \equiv \text{resistance}}$ (ohms (Ω))

$V_C \propto$ electric charge on the capacitor

$$U_C = \frac{1}{C} q$$

$\boxed{C \equiv \text{capacitance}}$ (farad (F))

Combining with $(*)$ yields

first order linear!

$$RI + \frac{1}{C} q = U(t)$$

then recall that $I = dq/dt$ so that $\boxed{R \frac{dq}{dt} + \frac{1}{C} q = U(t).}$

the initial condition is the charge at $t=0$.

Example

Suppose $V(t) = V$, a constant and $Q = Q_0$ at $t=0$ then

$$R\frac{dq}{dt} + \frac{1}{C}q = V$$ so that

$$R\frac{dq}{dt} = V - \frac{q}{C} = \frac{VC-q}{C} \Rightarrow \frac{dq}{dt} = \frac{VC-q}{RC}$$ is separable

$$\int \frac{dq}{VC-q} = \int \frac{1}{RC}\,dt$$

$$-\ln|VC-q| = \frac{1}{RC}t + K$$

$$VC - q = e^{-\left(\frac{1}{RC}t + K\right)} = e^{-\frac{1}{RC}t} \cdot e^{-K}$$

Applying the initial condition, $t=0$, $q=Q$

$$VC - Q_0 = e^{-K}$$ so that

$$VC - q = e^{-\frac{t}{RC}}(VC - Q_0)$$

$$\boxed{VC - (VC - Q_0)e^{-\frac{t}{RC}} = q(t)}$$

$q(t)$, the capacitor charge increases from Q to VC as t ~~now~~ increases.

FACT RI circuit

$$L\frac{dI}{dt} + RI = V(t)$$ (first order linear ODE).

L is the inductance.