**Fact I** (first order linear equations) (Existence and Uniqueness).

Given \( \frac{dy}{dt} + p(t)y = g(t) \), \( y(t_0) = y_0 \).

If \( p(t) \) and \( g(t) \) are continuous on an interval containing \( t_0 \), then there exists a unique solution to the first order linear ode. The solution can be found using the method of integrating factors.

**Note**

The solution may fail to exist or be discontinuous at points where \( p(t) \) or \( g(t) \) are discontinuous.

**Fact II** (first order non-linear equations) (Existence and uniqueness)

Given \( \frac{dy}{dt} = f(t,y) \), \( y(t_0) = y_0 \).

Let \( f \), \( \frac{df}{dy} \) be continuous in a rectangle, \( R \) containing \( (t_0, y_0) \).

Then there exists a unique solution to the ODE in some interval contained in \( \alpha < t < \beta \).

**Examples**

**Linear case**

Determine (without solving) the interval on which the solution to the IVP

\[
(\ln 2) y' + (\ln t)y = 2t \quad , \quad y(1) = 2
\]

is certain to exist.
In standard form:

\[ y' + \frac{\ln(t)}{t-3} \, y = \frac{2t}{t-3}, \quad y(1) = 2. \]

\[ p(t) = \frac{\ln(t)}{t-3} \quad \text{and} \quad g(t) = \frac{2t}{t-3} \]

\( p(t) \) is continuous on \((0,3) \cup (3,\infty)\) and \( g(t) \) on \((-\infty,3) \cup (3,\infty)\). So both \( p(t) \) and \( g(t) \) are defined and continuous on \((0,2) \cup (2,\infty)\). The initial point is \( t=1 \), so the solution will be continuous on \( 0 < t < 2 \).

**Nonlinear case**

\[ \frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \quad , \quad y(0) = -1. \]

\[ f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)} \]

\[ \frac{df}{dy} = \frac{(3x^2 + 4x + 2)(-1)(y-1)^{-2}}{2(y-1)^2} \]

\[ = \frac{3x^2 + 4x + 2}{2(y-1)^2} \]

\( f(x,y) \) and \( \frac{df}{dy} \) are continuous everywhere except \( y = 1 \).

**FACT 2** guarantees that the IVP has a unique solution in some region around \((0,-1)\)

\[ \frac{dy}{dx} = \frac{2x}{1+2y}, \quad y(2) = 0 \]

\[ f(x,y) = \frac{2x}{1+2y} \quad \text{and} \quad \frac{df}{dy} = \frac{-2x}{(1+2y)^2} \cdot 2 = -\frac{4x}{(1+2y)^2} \] has a unique solution around \((2,0)\).
Bernoulli Equations

\[ y' + p(t) y = q(t) y^n \quad (*) \]

For \( n \neq 0, 1 \)

Let \( u = y^{1-n} \), then

\[ \frac{du}{dt} = (1-n) y^{(1-n)-1} \frac{dy}{dt} = (1-n) y^{-n} \frac{dy}{dt} \]

Substitute into (*)

\[ \frac{1}{1-n} \cdot y^n \frac{du}{dt} + p(t) y = q(t) y^n \]

Integrate both sides to solve for \( u \)

\[ \frac{du}{dt} + (1-n) y^{1-n} = q(t) (1-n) \]

and recall that \( u = y^{1-n} \) so

\[ \frac{du}{dt} + (1-n) u = q(t) (1-n) \]

This is now linear in \( u \) so we can use the method of integrating factors!

Example

Solve \( y' = ry - ky^2 \)

\[ y - ry = -ky^2 \]

fits the generic form \( y' + p(t) y = q(t) y^n \). \( n = 2 \).

Let \( u = y^{-1} \) and so that \( \frac{du}{dt} = -y^{-2} \frac{dy}{dt} \) \( \Rightarrow -y^2 \frac{du}{dt} = \frac{dy}{dt} \)

\[ -y^2 \frac{du}{dt} - ry = -ky^2 \]

multiply out by \(-y^{-2}\) \[ u + ru = k \]

\[ \frac{du}{dt} + ru^{-1} = k \]

and make \( \mu(t) = e^{\int r dt} = e^{rt} \)

\[ (u' + ru) e^{rt} = ke^{rt} \]
\[ \frac{d}{dt} (e^{rt} v) = ke^{rt} \] and integrating on both sides,

\[ e^{rt} v = k \int e^{rt} = \frac{ke^{rt}}{r} + C \]

\[ v = e^{-rt} \left( \frac{ke^{rt}}{r} + C \right) = \frac{k}{r} + Ce^{-rt} \]

Finally, recall that \( v = \frac{1}{y} \) so

\[ \frac{1}{y} = \frac{k}{r} + Ce^{-rt} \]

**Discontinuous coefficients**

Solve

\[ y' + 2y = g(t), \quad y(0) = 0 \]

\[ g(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}. \]

\( g(t) \) is discontinuous at \( t = 1 \) so the derivative has a jump at \( t = 1 \) but it exists!

This means that \( y(t) \) has to be continuous at \( t = 1 \).

1. Solve IVP on \([0, 1]\)

\[ y' + 2y = 1, \quad \mu(t) = e^t = e^t \]

\[ (y' + 2y)e^{2t} = 1 \cdot e^{2t} \]

\[ \frac{d}{dt} (e^{2t} y) = e^{2t} \Rightarrow e^{2t} y_1 = \int e^{2t} dt = \frac{e^{2t}}{2} + C. \quad y(0) = 0 \quad \text{so} \quad C = \frac{1}{2} \]

2. Solve IVP on \((1, \infty)\)

\[ y'' + 2y = 0, \quad \mu(t) = e^{-2t} \]

\[ \frac{d}{dt} \left( e^{-2t} y_2 \right) = 0 \Rightarrow e^{-2t} y_2 = C \quad y_2 = Ce^{-2t} \]

Find \( C \) in \( y_2 \) so that \( y_2(1) = y_1(1) \)

\[ y_1(1) = \frac{1}{2} - \frac{1}{2} e^{-2} = \frac{1}{2} (1 - e^{-2}) \]

\[ y_2(1) = \frac{1}{2} - \frac{1}{2} e^{-2} = \frac{1}{2} (1 - e^{-2}) \]

\[ C = \frac{e^2}{2} (1 - e^{-2}) = \frac{1}{2} (e^2 - 1) \]
\[ y(t) = \begin{cases} \frac{1}{2} (1 - e^{-2t}) , & 0 \leq t \leq 1 \\ \frac{1}{2} (e^t - 1) e^{-2t} , & t > 1 \end{cases} \]

Note that
\[ y'(t) = \begin{cases} 2e^{-2t} , & 0 \leq t \leq 1 \\ \frac{1}{2} (e^t - 1)(-2e^{-2t}) , & t > 1 \end{cases} \]

is discontinuous at \( t = 1 \).