2.4

Existence and Uniqueness of Solutions

FACT I (first order linear equations) (Existence and Uniqueness).

Given $\frac{dy}{dt} + p(t)y = g(t)$, $y(t_0) = y_0$

If $p(t)$ and $g(t)$ are continuous on an interval containg t_0, then there exists a unique solution to the first order linear ode.

The solution can be found using the method of integrating factors.

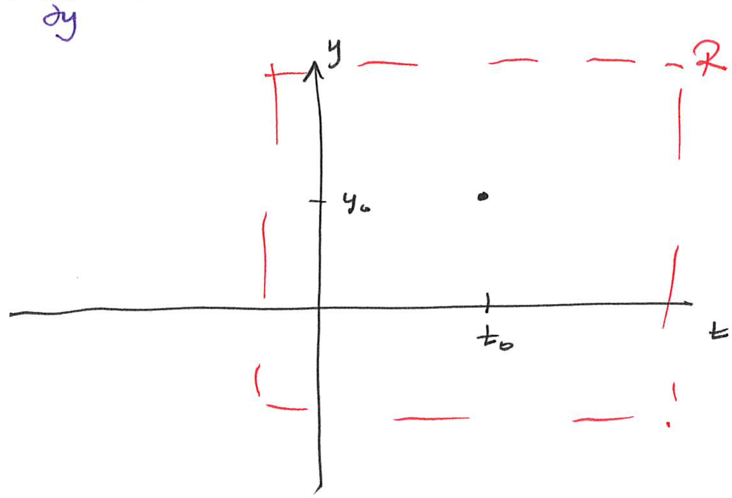
Note

The solution may fail to exist OR be discontinuous at points when $p(t)$ OR $g(t)$ are discontinuous.

FACT II (first order non-linear equations) (Existence and uniqueness)

Given $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$

Let f, $\frac{\partial f}{\partial y}$ be continuous in a rectangle, R containing (t_0, y_0)



$\alpha \leq t \leq \beta$
$\gamma < y < \delta$

then there exists a unique solution to the ODE in some interval contained in $\alpha < t < \beta$.

Examples.

Linear case

Determine (without solving) the interval on which the solution to the IVP

$$(t-3)y' + (\ln t)y = 2t \quad , \quad y(1) = 2$$

is certain to exist.

In standard form :

$$y' + \frac{\ln(t)}{t-3}\, y = \frac{2t}{t-3}, \qquad y(1) = 2.$$

$$p(t) = \frac{\ln(t)}{t-3} \quad \text{and} \quad g(t) = \frac{2t}{t-3}$$

$p(t)$ is continuous on $(0,3) \cup (3,\infty)$ and $g(t)$ on $(-\infty,3) \cup (3,\infty)$. So both $p(t)$ and $g(t)$ are defined and continuous on $(0,3) \cup (3,\infty)$. The initial point is $t=1$, so the solution will be continuous on $0 < t < 3$.

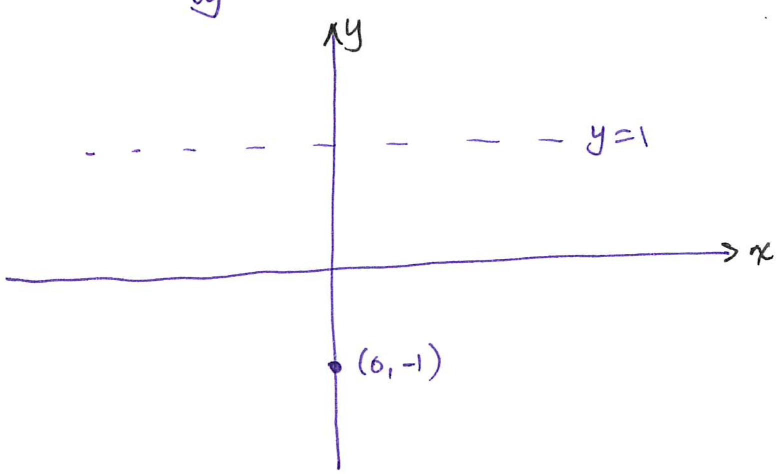
Nonlinear case

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1.$$

$$f(x,y) = \frac{3x^2+4x+2}{2(y-1)} \qquad \frac{\partial f}{\partial y} = (3x^2+4x+2)\left(-\frac{1}{2}\right)(y-1)^{-2}$$

$$= (3x^2+4x+2)\cdot \frac{1}{2}\,\cancel{y-1)}\,(y-1)^{-1} \qquad = -\frac{(3x^2+4x+2)}{2(y-1)^2}$$

$f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous everywhere except $y=1$.



FACT II gaurantees that the IVP has a unique solution in some region around $(0,-1)$

$$\frac{dy}{dx} = \frac{2x}{1+2y}, \quad y(2) = 0$$

$$f(x,y) = \frac{2x}{1+2y}, \quad \frac{\partial f}{\partial y} = \frac{-2x}{(1+2y)^2}\cdot 2 = -\frac{4x}{(1+2y)^2}$$ has a unique solution around $(2,0)$

Bernoulli Equations

$$y' + p(t)\,y = q(t)\,y^n \quad (*)$$

$n = 0, 1$ (linear case)

$n \neq 0, 1$

$\boxed{\text{Let } v = y^{1-n}}$, then $\dfrac{dv}{dt} = (1-n)\,y^{(1-n)-1}\cdot\dfrac{dy}{dt} = (1-n)\,y^{-n}\cdot\dfrac{dy}{dt}$

Substitute into $(*)$

$$\underbrace{\frac{1}{1-n}\cdot y^n\,\frac{dv}{dt}}_{\frac{dy}{dt}} + p(t)\,y = q(t)\,y^n$$

to write every term in terms of v and the ODE in the general form multiply by $(1-n)\,y^{-n}$

$$\frac{dv}{dt} + (1-n)\,y^{1-n} = q(t)\,(1-n)$$

and recall that $v = y^{1-n}$ so that

$$\boxed{\frac{dv}{dt} + (1-n)\,v = q(t)\,(1-n)}$$

This is now linear in v so we can use the method of integrating factors!

Example

Solve $y' = ry - ky^2$

$\boxed{y' - ry = -ky^2 \text{ fits the generic form } y' + p(t)\,y = q(t)\,y^n. \quad n = 2.}$

let $v = y^{-1}$ ~~and~~ so that $\dfrac{dv}{dt} = -y^{-2}\cdot\dfrac{dy}{dt}$. $\Rightarrow -y^2\,\dfrac{dv}{dt} = \dfrac{dy}{dt}$

$$-y^2\,\frac{dv}{dt} - ry = -ky^2$$

$$\frac{dv}{dt} + rv = k$$

multiply out by $-y^{-2}$

$$\frac{dv}{dt} + r\,y^{-1} = k \quad \text{and let } \mu(t) = e^{\int r\,dt} = e^{rt}$$

$$\left(v' + \cancel{ry^{-1}}\,rv\right)\cdot e^{rt} = k\,e^{rt}$$

$$\frac{d}{dt}\left(e^{rt}v\right) = ke^{rt}$$ and integrating on both sides

$$e^{rt}v = k\int e^{rt} = \frac{ke^{rt}}{r} + C$$

so $$v = e^{-rt}\left(\frac{ke^{rt}}{r} + c\right) = \frac{k}{r} + ce^{-rt}$$

Finaly, recall that $v = \frac{1}{y}$ so

$$\frac{1}{y} = \frac{k}{r} + ce^{-rt} \Rightarrow \boxed{y = \frac{1}{\frac{k}{r} + ce^{-rt}}}$$

Discontinuous coefficients

Solve $$y' + 2y = g(t), \quad y(0) = 0$$

$$g(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0 & t > 1. \end{cases}$$

$g(t)$ is discontinuous @ $t=1$ so the derivative has a jump @ $t=1$ but it exists!
This means that $y(t)$ has to be continuous at $t=1$.

① Solve IVP on $[0,1]$

$$y_1' + 2y_1 = 1 \qquad \mu(t) = e^{\int 2\,dt} = e^{2t}$$

$$(y_1' + 2y_1)e^{2t} = 1 \cdot e^{2t}$$

$$\frac{d}{dt}\left(e^{2t} \cdot y_1\right) = e^{2t} \Rightarrow e^{2t}y_1 = \int e^{2t}dt = \frac{e^{2t}}{2} + C. \quad y(0) = 0 \text{ so } C = -\frac{1}{2}$$

$$y_1(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

② Solve IVP on $(1, \infty)$

$$y_2' + 2y_2 = 0 \qquad \mu(t) = e^{-2t}$$

$$\frac{d}{dt}\left(e^{2t}y_2\right) = 0 \Rightarrow e^{2t}y_2 = C \qquad y_2 = Ce^{-2t}$$

Find C, in y_2 so that $y_2(1) = y_1(1)$

$$y_1(1) = \frac{1}{2} - \frac{1}{2}e^{-2} = \cancel{\tfrac{1}{2}(1-e^{-2})} = Ce^{-2} \Rightarrow \cancel{\tfrac{1}{2} = Ce^{-2}} + \tfrac{1}{2}e^{-2}$$

$$C = \frac{e^2}{2}(1 - e^{-2}) = \frac{1}{2}(e^2 - 1)$$

$$y(t) = \begin{cases} \frac{1}{2}\left(1-e^{-2t}\right), & 0 \le t \le 1 \\ \frac{1}{2}(e^2-1)\,e^{-2t}, & t > 1 \end{cases}$$

Notice that

$$y'(t) = \begin{cases} 2e^{-2t}, & 0 \le t \le 1 \\ \frac{1}{2}(e^2-1)\left(-2e^{-2t}\right), & t > 1 \end{cases}$$ is discontinuous at $\underline{t=1}$.