2.5 Autonomus Equations

$y' = f(y)$ i.e the slope field depends only on y (not explicitly on t)

Examples

(a) $y' = y^2$ and $P' = rP(1-P)$ are autonomous $\left(P' = 0.4P\left(1 - \frac{P}{100}\right) - H\right)$ from homework #1.

(b) $y' = y + t$, $y' = y^2 - t$ are not.

Solutions of Autonomous Equations (Behavior)

Equilibrium solutions

If $f(r) = 0$, then $y(t) = r$ is an equilibrium solution

Example

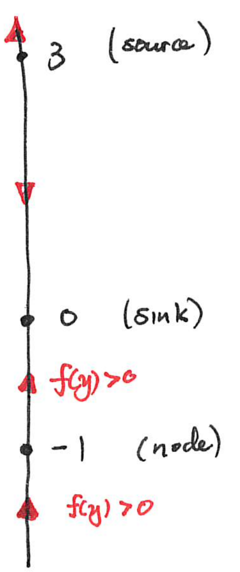
$$y' = y(y+1)^2(y-3) = f(y)$$

Equilibrium solutions

zeros of $f(y)$ i.e $y = 0, -1, 3$

Phase line

Plot showing equilibrium solution and the sign of y'



$y' = y(y+1)^2(y-3)$

$y < -1$, $y' > 0$

Pick any $y < -1$, e.g $y = -2$ plug into y'

$0 < y < -1$, $y' > 0$

$0 < y < 3$, $y' < 0$

$y > 3$, $y' > 0$

Remarks

1. If $y(0) < -1$, $y' > 0$ so $y(t)$ is increasing. However $y(t)$ cannot cross the $y(t) = -1$ line due to uniqueness (FACT II). This means $y(t) \to -1$ as $t \to \infty$.

Stability

1. Sinks

If $y(0)$ is close to 0, $y(t) \to 0$ (from both sides)

$y=0$ is called a SINK and is said to be a STABLE equilibrium / asymptotically stable

2. If $y(0)$ is close to $y=3$, $y(t)$ moves away from 3

$y=3$ is called a SOURCE and is said to be an UNSTABLE equilibrium,

3. For solutions close to $y=-1$, $y(0) < -1$, $y \to -1$ but for $y(0) > -1$, $y(t)$ diverges away. ~~$y=-1$ is called a NOD~~.

$y=-1$ is called a NODE and is said to be a SEMI-STABLE equilibrium.

Matlab attachment

Bifurcation in ODEs with parameters

$y' = f(y, \alpha)$, α is a parameter.

Birfurcation

A small smooth change in parameter results in a sudden qualitative change in the solution.

Examples
1. Smooth flowing fluid suddenly becomes turbulent
2. loaded column may suddenly buckle.

Example

Show that $y' = f(y,\alpha) = \alpha y - y^3$ bifurcates when α passes through 0.

for $\alpha \le 0$

$\alpha y - y^3 = y(\alpha - y^2)$ has one real solution $y=0$ (One equilibrium sol)

$\alpha > 0$

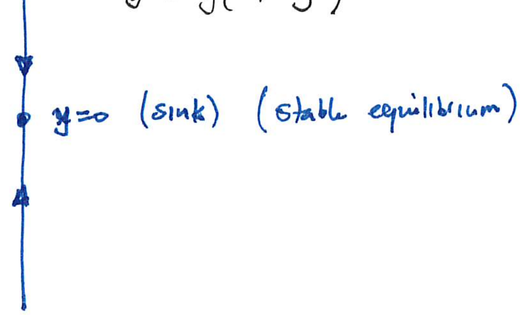
$y(\alpha - y^2)$ has 3 solutions $y=0$, $y = \pm\sqrt{\alpha}$ (3 equilibrium sols)

e.g $\alpha = -1$

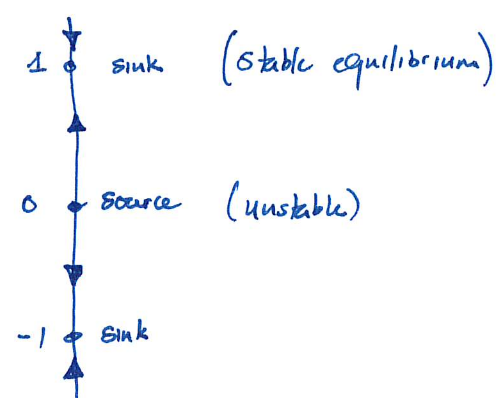
Phase diagram

$y' = \cancel{\alpha y} \cdot y(\alpha - y^2)$

$y' = y(-1-y^2)$



$\alpha = 1$ $\quad y' = y(1-y^2)$



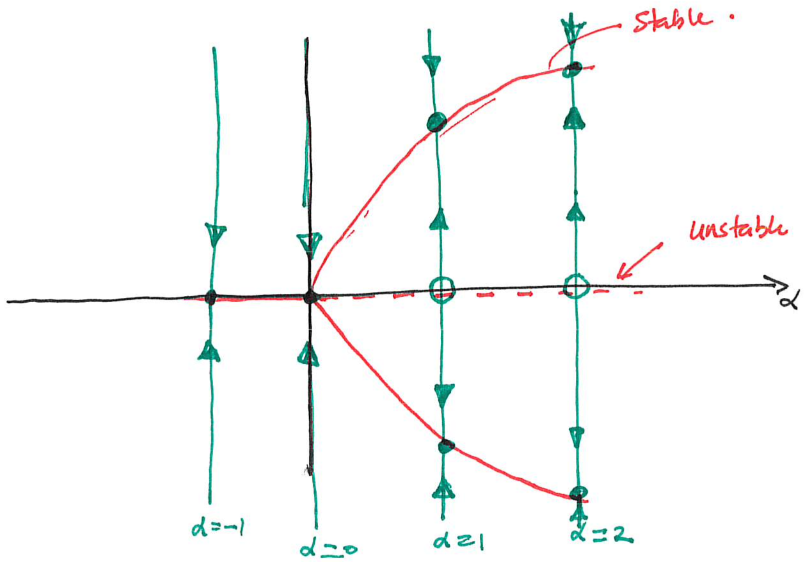
Birfurcation diagram

* Collection of phase lines close to the bifurcation value.

Notation

Stable equilibria (Sinks) are connected by solid curves

Unstable equilibria (sources) are connected by dotted lines



$\alpha = 2$

$y' = y(2 - y^2)$

$y = 0, \quad y = \pm\sqrt{2}$

Pitch fork bifurcation:

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{100}\right) - H$$

This is the Harvested Logistic Equation from hw #1.

Recall that there was a change in the solution behavior between $H=6$ and 12.

$f(P, H) = rP\left(1 - \frac{P}{N}\right) - H$ so the equilibrium solutions are values of P

s.t. $f(P, H) = 0$

$$rP - \frac{rP^2}{N} - H = 0 \Rightarrow -\frac{r}{N}P^2 + rP - H = 0 = -\frac{r}{N}\left(P^2 - NP + \frac{NH}{r}\right)$$

Solving $P^* = \dfrac{N \pm \sqrt{N^2 - \left(4 \cdot 1 \cdot \frac{NH}{r}\right)}}{2} = \dfrac{N}{2} \pm \dfrac{N}{2}\sqrt{1 - \left(\frac{4}{rN}\right)H}$

Case I

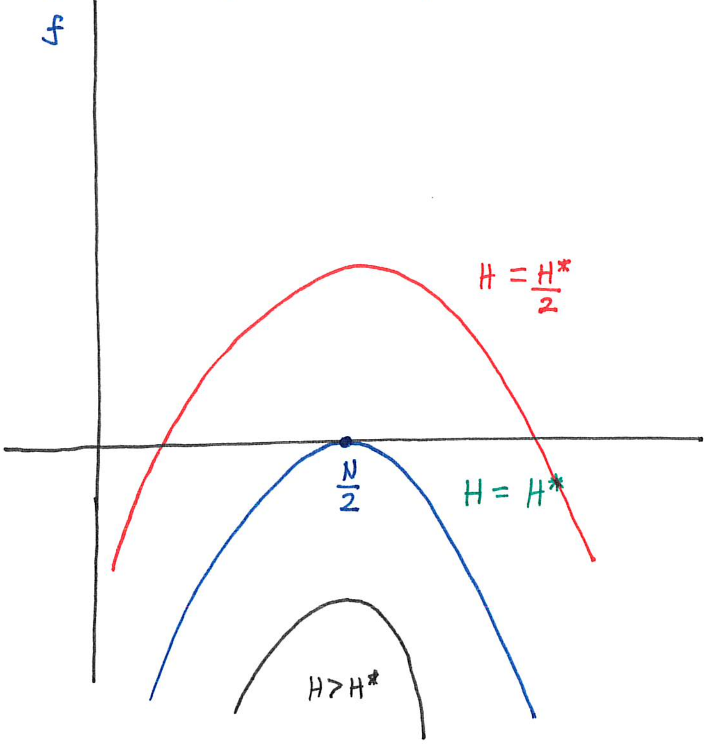
If $1 - \frac{4H}{rN} = 0$ we have one equilibrium solution $P = \frac{N}{2}$

$\Downarrow \quad 1 - \frac{4H}{rN} = 0 \quad \frac{rN}{4} = H$

$$\boxed{H^* = \frac{rN}{4}}$$

for $1 - \frac{4H}{rN} < 0$, no solutions and $1 - \frac{4H}{rN} > 0$ we have 2 solutions.

$f(P,H) = rP\left(1-\frac{P}{N}\right) - H$



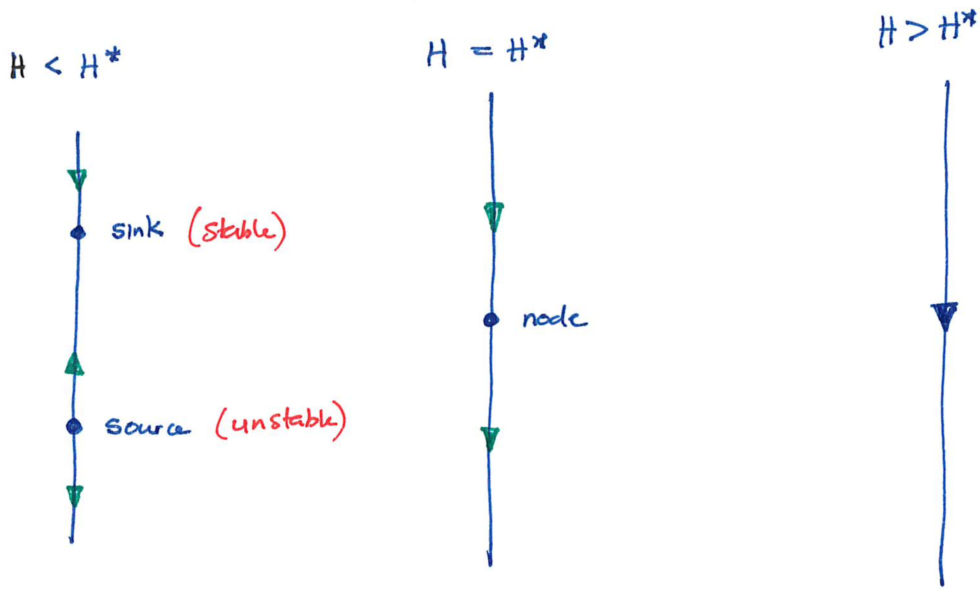
$1 - \frac{4H}{rN} < 0$

$1 < \frac{4H}{rN}$ $\quad H > \frac{rN}{4}$

$H > H^*$

$1 - \frac{4H}{rN} > 0 \Rightarrow H < H^*$

2 sols



In our problem $N = 100$ $\quad r = 0.4$ $\quad H^* = \frac{rN}{4} = \frac{0.4(100)}{4} = 10.$

$H < 10$ we have 2 equilibrium solutions 65.8 and 84.2
stable unstable

<u>$H = 10$</u>

Node (semi-stable) equilibrium

<u>$H = 12$</u> extinction regardless of starting value.

$H = H^*$ is called the critical harvesting rate.