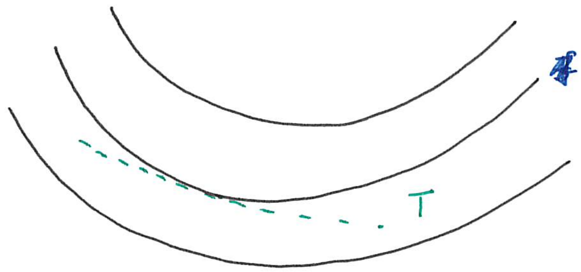
2.6

Exact ODEs

Recall that $\psi(x,y) = c$ is the level curve of a surface, $z = f(x,y)$, where $y = f(x)$.



$\psi(x,y) = c$ e.g. isotherms on a temperature map.

Q What is the slope of the tangent line T. ?

We need the derivative! (Implicit!) - - - - → $\psi(x,y)$

$$\begin{array}{ccc} & \psi(x,y) & \\ x & & y \\ & & | \\ & & x \end{array}$$

$$\frac{d}{dx}\left(\psi(x,y)\right) = \frac{d}{dx}(c)$$

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \cdot \frac{dy}{dx} = 0 \quad (**)$$ and solving for $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}}} \quad (*)$$

(*) has the form of an ODE (first order). What is a possible solution? $\psi(x,y) = c$!

Total differential, $d\psi$

from (**), mutiplying by dx yeilds

$$\underbrace{\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy}_{} = 0$$

The left hand side is called the ~~de~~ total differential.

Objective Solve ODEs of the form

$$\boxed{\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0}$$

Side Note Any first order ODE of the form $\frac{dy}{dx} = f(x,y)$ can be written in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$ Check

$$-f(x,y) + \frac{dy}{dx} = 0 \Rightarrow$$

$$-f(x,y)\,dx + dy = 0$$

Example #2

Solve

$$(2xy - \sec^2(x))\,dx + (x^2 + 2y)\,dy = 0$$

Test

$M(x,y) = 2xy - \sec^2(x)$ $\qquad$ $N(x,y) = (x^2+2y)$

$M_y(x,y) = 2x$

$N_x(x,y) = 2x$ ✓

ODE is exact! i.e it has the form

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0.$$

Find $\psi(x,y)$

Since $M(x,y) = \psi_x(x,y) = 2xy - \sec^2(x)$, integrate on both sides w.r.t x

so that

$$\psi(x,y) = \int (2xy - \sec^2(x))\,dx = \frac{2x^2}{2}y - \tan(x) + h(y) = x^2y - \tan(x) + h(y)$$

$$\frac{d}{dy}\left(\psi(x,y)\right) = x^2 - 0 + h'(y) = N(x,y) = x^2 + 2y$$

So we have

$$x^2 + h'(y) = x^2 + 2y \Rightarrow h'(y) = 2y \Rightarrow h(y) = \int 2y\,dy$$
$$= \frac{2y^2}{2} + c$$
$$= y^2 + c$$

Here we can ignore the constant and write

$$\psi(x,y) = x^2y - \tan(x) + y^2$$

so that our general implicit solution is

$$\psi(x,y) = c \quad \text{or} \quad \boxed{x^2y - \tan(x) + y^2 = C}$$

but our interest is in the ones which have a rhs that takes the form of a total differential i.e

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy}$$

← This is called an EXACT equation.

so that solving the ODE means finding $\psi(x,y)$, and the solutions are $\psi(x,y) = c$.

Example

Solve $\quad \dfrac{dy}{dx} = -\dfrac{2xy^2+1}{2x^2y}.$

Note, the ODE is not separable.

$$2x^2y\,dy = -(2xy^2+1)\,dx \quad \Rightarrow \quad (2xy^2+1)\,dx + 2x^2y\,dy = 0$$

Observe that

$$\frac{d}{dx}\left(x^2y^2+x\right) = 2xy^2+1 \quad \text{and} \quad \frac{\partial}{\partial y}\left(x^2y^2+x\right) = 2x^2y \quad \text{so}$$

the general solution is given by

$$\boxed{x^2y^2 + x = c}$$

TEST for EXACT Equations

If $M(x,y)$ and $N(x,y)$ are continuous and $M(x,y) = \dfrac{\partial \psi}{\partial x}$ and $N(x,y) = \dfrac{\partial \psi}{\partial y}$ then Clairaut's theorem (Calc III) on mixed partials says that if the mixed partial derivatives of ψ are continuous then

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)$$

FACT (Test)

Suppose $M(x,y)$ and $N(x,y)$ are continuous, then

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial}{\partial y}\left(M(x,y)\right) = \frac{\partial}{\partial x}\left(N(x,y)\right)$$

How to find $\psi(x,y)$ in general?

Back to $\frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y} \Leftrightarrow (2xy^2+1)dx + 2x^2y\,dy = 0$

TEST

$M(x,y) = 2xy^2+1 \Rightarrow \cancel{M_y}\ M_y(x,y) = \frac{d}{dy}(2xy^2+1) = 4xy$

$N(x,y) = 2x^2y \Rightarrow N_x(x,y) = \frac{d}{dx}(2x^2y) = 4xy$

So the ODE is exact ✓

FIND $\psi(x,y)$ such that

$\psi_x(x,y) = M(x,y) = 2xy^2+1 \quad \ldots (i)$

$\psi_y(x,y) = N(x,y) = 2x^2y. \quad \ldots (ii)$

1. Integrate (i) w.r.t x yeilds

$\psi(x,y) = \int (2xy^2+1)dx = \frac{2x^2}{2}y^2 + x + h(y)$ this is our arbitrary constant.

2. $\psi_y(x,y) = \frac{d}{dy}\left(x^2y^2 + x + h(y)\right) = 2x^2y$

$x^2 2y + 0 + h'(y) = 2x^2y \Rightarrow h'(y) = 0$

So $h(y) = c.$

So our solution is $\psi(x,y) = x^2y^2 + x = c.$

Check $\psi(x,y) = c \Rightarrow \cancel{x^2y}\ x^2y^2+x = c$ satisfies ODE, just compute y'

$\frac{d}{dx}\left(x^2y^2+x\right) = 0$

$\frac{dy}{dx} = \frac{-\psi_x}{\psi_y} = -\frac{2xy^2+1}{x^2 2y}$

Integrating Factor

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, introduce an integrating factor $\mu(x,y)$ that forces

$$\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$$ to be exact.

So we need

$$\frac{\partial}{\partial y}\left[\mu(x,y)\,M(x,y)\right] = \frac{\partial}{\partial x}\left[\mu(x,y)\,N(x,y)\right]$$

Product Rule

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$ so that

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu$$ (This, however is a Pde and equaly difficult if not harder to solve)

Lets make some simplifications

say $\mu = \mu(x)$, then $\frac{\partial \mu}{\partial y} = 0$ and $\frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$ so

$$\frac{d\mu}{dx} = \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{-N}\right)\mu = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right)\mu$$ which is seperable provided

$$\left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right)$$ is only a function of x.

or

$\mu = \mu(y)$, then $\frac{\partial \mu}{\partial y} = \frac{d\mu}{dx}$ and $\frac{\partial \mu}{\partial x} = 0$

$$\frac{d\mu}{dy} = \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right)\mu$$ separable provided

$$\left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right)$$ is a function only of y.

In each of the simplified cases

$$\mu(x) = e^{\int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right) dx}$$

$$\mu(y) = e^{\int \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right) dy}$$

<u>Example #1</u>

Solve

$$(2x^2 + y)\,dx + (x^2 y - x)\,dy = 0 \qquad \left(M(x,y)\,dx + N(x,y)\,dy = 0 \right)$$

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2xy - 1$$ so the ODE is not exact.

Search for an integrating factor in x or y.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1 - (2xy - 1)}{x^2 y - x} = \frac{2 - 2xy}{x^2 y - x} = \frac{2(1 - xy)}{-x(1 - xy)} = \frac{-2}{x}.$$

$$x(xy - 1) = -x(1 - xy)$$

$$\mu(x) = e^{\int -\frac{2}{x} dx} = e^{-2\ln|x|} = e^{-\ln(x^2)} = e^{\ln(x^{-2})} = \underline{x^{-2}}$$

Multiply the ODE by $\mu(x) = x^{-2}$.

$$(2x^2 + y)(x^{-2})\,dx + (x^2 y - x)(x^{-2})\,dy = 0$$

$$(2 + yx^{-2})\,dx + (y - x^{-1})\,dy = 0$$

Check that this is exact

$$M_y(x,y) = x^{-2}$$
$$N_x(x,y) = x^{-2}$$ ✓

Recalling that

$$\psi_x(x,y) = 2 + yx^{-2}$$

Integrate on both sides w.r.t x

$$\psi(x,y) = 2x + y(-x^{-1}) + h(y)$$

$\psi_y(x,y) = \frac{-1}{x} + h'(y) = N(x,y) = y - \frac{1}{x}$

$\Rightarrow \quad h'(y) = y \quad$ so $\quad h(y) = \int y\,dy = \frac{y^2}{2} + C$

So $\psi(x,y) = 2x - \frac{y}{x} + \frac{y^2}{2}$ and thus our general solution is

$$2x - \frac{y}{x} + \frac{y^2}{2} = C. \quad (*)$$

Note ① We lost the $x=0$ solution when we multiplied by $\mu = \frac{1}{x^2}$.

This is also a solution (in addition to $(*)$).

② If we had tried an integrating factor in y, then we compute

$$\mu \cdot \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(2xy-1)-1}{2x^2+y} = \frac{2xy-2}{2x^2+y} \quad \text{so}$$

an integrating factor only in y cannot be found.