3.1 Second - order linear ODEs

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Linear case

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y.$$

i.e f is linear in both y and $\frac{dy}{dt}$, t is an independent variable.

In short $y'' = g(t) - p(t)y' - q(t)y$ or

$$\boxed{y'' + p(t)y' + q(t)y = g(t)}$$

Remarks

* Second order non-linear equations are very difficult to solve (we will see them at the end of the semester)

** We will ~~assume~~ consider 2^{nd} order linear ODEs on intervals in which $p(t)$, $q(t)$ and $g(t)$ are continuous.

Initial conditions

In general, we need 2

eg $y(t_0) = y_0$
$y'(t_0) = y_0'$ e.g initial position and velocity.

Homogeneous Equations

If $g(t) = 0$ for all t, then

$$\boxed{y'' + p(t)y' + q(t)y = 0}$$

is called a 2^{nd} order linear homogeneous equation otherwise it is called Nonhomogeneous.

Will will focus on equations of the form

$$ay'' + by' + cy = 0$$

to start.

Example #1.

$$y'' + 2y' - y = 0 \quad , y(0) = 0 \quad , y'(0) = 1.$$

The characteristic equation is

$$r^2 + 2r - 1 = 0$$

$$r = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} \qquad -1 \pm \sqrt{2}.$$

$$\boxed{y(t) = c_1 e^{(-1+\sqrt{2})t} + c_2 e^{(-1-\sqrt{2})t}} \qquad \text{(General solution).}$$

Specific solution satisfies $y(0) = 0$ and $y'(0) = 1$

$$y(0) = 0 \quad \Rightarrow \quad c_1 + c_2 = 0 \quad \ldots \quad (i)$$

$$y'(t) = (-1+\sqrt{2})\, c_1 e^{(-1+\sqrt{2})t} + (-1-\sqrt{2})\, c_2 e^{(-1-\sqrt{2})t}$$

$$y'(0) = 1 \quad \Rightarrow \quad (-1+\sqrt{2})\, c_1 \cdot e^0 + (-1-\sqrt{2}) c_2 e^0 = 1$$

$$(-1+\sqrt{2})\, c_1 + (-1-\sqrt{2})\, c_2 = 1 \quad \ldots\ldots \quad (ii)$$

from (i) $c_1 = -c_2$, plug into (ii)

$$(-1+\sqrt{2})(-c_2) + (-1-\sqrt{2})\, c_2 = 1$$

$$c_2 - \cancel{2}\sqrt{2}\, c_2 - c_2 - \cancel{2}\sqrt{2}\, c_2 = 1$$

$$-2\sqrt{2}\, c_2 = 1 \quad \Rightarrow \quad c_2 = -\frac{1}{2\sqrt{2}} \qquad c_1 = -c_2 = +\frac{1}{2\sqrt{2}}$$

$$\boxed{y(t) = +\frac{1}{2\sqrt{2}} e^{(-1+\sqrt{2})t} - \frac{1}{2\sqrt{2}} e^{(-1-\sqrt{2})t}}$$

Objective

Solve
$$ay'' + by' + cy = 0 \qquad (*)$$

We need a function whose second derivative can be expressed as a linear combination of y' and y:

$$\boxed{\begin{aligned} y(t) &= e^{rt} \\ y' &= re^{rt}, \quad y'' = r^2 e^{rt} \end{aligned}}$$

Plug into $(*)$

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$e^{rt}(ar^2 + br + c) = 0, \quad \text{since} \quad e^{rt} \neq 0 \quad ar^2 + br + c = 0$$

* $ar^2 + br + c = 0$ is called the characteristic equation of the ODE.

Solving $ar^2 + br + c = 0$

Case I: Distinct real roots (i.e $b^2 - 4ac > 0$) (r_1 and r_2)

We thus have

$y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$

FACT

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is also a solution to $ay'' + by' + cy = 0$

Check

$$y''(t) = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}$$
$$y'(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$$

Plug into ODE

$$a(c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}) + b(c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}) + c(c_1 e^{r_1 t} + c_2 e^{r_2 t})$$

$$c_1(ar_1^2 + br_1 + c)e^{r_1 t} + c_2(ar_2^2 + br_2 + c)e^{r_2 t} = 0$$

Since both r_1 and r_2 are roots

$$(ar_1^2 + br_1 + c) = 0 \quad \text{and} \quad ar_2^2 + br_2 + c = 0 \quad \text{so}$$

Note that $y(t) \equiv 0$ is also a solution to $(*)$.

$$\boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

is a general solution. We can find c_1 & c_2 from initial conditions.