3.2

Solutions to Linear ~~homogeneous~~ equations

FACT (Existence and uniqueness)

Given $y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0$ and $y'(t_0) = y_0'$ where $p(t)$, $q(t)$ and $g(t)$ are continuous for $t_1 < t < t_2$, then there exists a unique solution $y(t)$ that is continuous on (t_1, t_2).

Example

(a) Find the longest interval on which

$$t(t-4)y'' + 3t\,y' + 4y = 2 \quad , \quad y(3) = 0 \quad , \quad y'(3) = -1$$

has a unique solution.

$$y'' + \frac{3t}{t(t-4)}y' + \frac{4y}{t(t-4)} = \frac{2}{t(t-4)}$$

The coefficient functions are continuous on $(-\infty, 0) \cup (0,4) \cup (4,\infty)$

$t_0 = 3$ so $I = (0,4)$.

(b) What about the solution to

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0 \quad , \quad y'(t_0) = 0$$

If $p(t)$, $q(t)$ are assumed to be continuous in I containing t_0.

$y = \phi(t) = 0$ is the only solution!

Principle of superposition

Background : We can define the LHS of our ODE as

$$L[y] = y'' + p(t)y' + q(t)y$$

FACT

$L[y]$ is a linear operator i.e

$$L[f \pm g] = L[f] \pm L[g]$$

$$L[cf] = cL[f]$$

If y_1 and y_2 are solutions to $L[y] = 0$, then $c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2.

Q1 Given $y(t_0) = y_0$ and $y'(t_0) = y_0'$, can we always find c_1 and c_2?

To find c_1 and c_2, we need to ensute that $y(t_0) = y_0$ & $y'(t_0) = y_0'$ for $y(t) = c_1 y_1 + c_2 y_2$

$$y(t_0) = y_0 \Rightarrow c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \quad \cdots (i)$$

$$y'(t_0) = y_0' \Rightarrow c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \quad \cdots (ii)$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\vec{c}} = \underbrace{\begin{pmatrix} y_0 \\ y_0' \end{pmatrix}}_{\vec{r}}$$

(i) and (ii) has a unique solution if and only if $\det(A) \neq 0$

$$\det(A) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \underbrace{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}_{\text{This is called the WRONSKIAN of the ODE, hence}} = W(y_1, y_2)[t_0]$$

Ans to Q1

Fact

If y_1 and y_2 are any 2 solutions of $L[y] = 0$ and $W(y_1, y_2) \neq 0$ at $t = t_0$ then we can find c_1 and c_2.

Q2

Are all solutions to $L[y] = 0$ contained in the form $y(t) = c_1 y_1(t) + c_2 y_2(t)$?

YES, provided there exists a point t_0 where $W(y_1, y_2)[t_0] \neq 0$.

Notation

In this case, (non-zero Wronskian) $y = c_1 y_1(t) + c_2 y_2(t)$ is a general solution and $\{y_1, y_2\}$ is called the FUNDAMENTAL set of solutions.

Example

Given $y'' - y' - 2y = 0$. Verify that $y_1 = e^{-t}$ and $y_2 = e^{2t}$ form a fundamental set

Recall that the characteristic polynomial is $r^2 - r - 2 = 0 \Leftrightarrow (r+1)(r-2) = 0$

$r = -1, \quad r = 2$

so

$y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ are solutions to the ODE.

$$W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_1'(t) y_2(t)$$
$$= e^{-t}(2e^{2t}) - (-e^{-t})(e^{2t}) = 2e^t + e^t = 3e^t \neq 0, \text{ for all } t \text{ so}$$

$\{e^{-t}, e^{2t}\}$ is a fundamental set of solutions.

Complex version of superposition

KT

If $y(t) = u(t) + i v(t)$ solves $L[y] = 0$, then $u(t)$ and $v(t)$ are also solutions of $L[y] = 0$.

Computing the Wronskian without solving ODE

Given $L[y] = y'' + p(t) y' + q(t) y = 0$, with p, q continuous on $t_1 < t < t_2$,

then

$$W[y_1, y_2](t) = c e^{-\int p(t)\,dt}$$

where c depends on y_1 and y_2. This means that $W[y_1, y_2](t)$ is either zero for all t or unequal to zero at $\cdot$

Example

Back to $y'' - y' - 2y = 0$, $W[y_1, y_2](t) = c e^{-\int (-1)\,dt} = c e^{t}$.

We computed $3e^{t}$ so c (above) is 3.