3.3 Complex Roots

Recall that the characteristic polynomial of $\boxed{ay'' + by' + cy = 0}$ is $ar^2 + br + c = 0$

Case II $b^2 - 4ac < 0$

The roots are then $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ where $i = \sqrt{-1}$ and λ and μ are real :

$$\lambda = \frac{-b}{2a} \quad \text{and} \quad \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

$\circ\circ\circ \quad r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b}{2a} \pm i \dfrac{\sqrt{4ac - b^2}}{2a}$

The corresponding solutions are $y_1(t) = e^{(\lambda + i\mu)t}$ and $y_2(t) = e^{(\lambda - i\mu)t}$

The original ODE is given in terms of real coefficients – we would like to write the solution as such

FACT (Euler's Formula)

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Taking $y_1(t) = e^{(\lambda + i\mu)t} = e^{\lambda t} \cdot e^{i\mu t}$,

$$e^{i\mu t} = \cos(\mu t) + i\sin(\mu t) \quad \text{so that}$$

$$y_1(t) = e^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right)$$

$$\boxed{y_1(t) = e^{\lambda t}\cos(\mu t) + i e^{\lambda t}\sin(\mu t)}$$

Similarly

$$\boxed{y_2(t) = e^{\lambda t}\cos(\mu t) - i e^{\lambda t}\sin(\mu t)}$$

$y_1(t)$ and $y_2(t)$ have now been written as a linear combination of real valued functions.

FACT

$$\frac{d}{dt}\left(e^{rt}\right) = re^{rt} \quad \text{for complex valued } r.$$

This easily follows from Euler's formula.

Example #1

Find the general solution to

$$y'' + 2y' + 4y = 0$$

The characteristic polynomial is $r^2 + 2r + 4 = 0$ with roots

$$r = \frac{-2 \pm \sqrt{4 - (4.4)}}{2} = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm i\sqrt{12}}{2} = \frac{-2 \pm i\,2\sqrt{3}}{2} = \underline{-1 \pm i\sqrt{3}}.$$

$r_1 = -1 + i\sqrt{3}$ $\quad$ $r_2 = -1 - i\sqrt{3}$.

$$y_1(t) = e^{(-1+i\sqrt{3})t} = e^{-t} \cdot e^{i\sqrt{3}t} = e^{-t}\left(\cos(\sqrt{3}t) + i\sin(\sqrt{3}t)\right)$$

$$y_2(t) = e^{(-1-i\sqrt{3})t} = e^{-t} \cdot e^{-i\sqrt{3}t} = e^{-t}\left(\cos(-\sqrt{3}t) + i\sin(-\sqrt{3}t)\right)$$

$$= e^{-t}\left(\cos(\sqrt{3}t) - i\sin(\sqrt{3}t)\right).$$

Are y_1 and y_2 fundamental solutions!

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$$= e^{(-1+i\sqrt{3})t} \cdot (-1-i\sqrt{3})\, e^{(-1-i\sqrt{3})t} - \left((-1+i\sqrt{3})\, e^{(-1+i\sqrt{3})t} \cdot e^{(-1-i\sqrt{3})t}\right)$$

$$= \left((-1-i\sqrt{3})\, e^{-2t}\right) - \left((-1+i\sqrt{3})\, e^{-2t}\right)$$

$$= -2i\sqrt{3}\, e^{-2t} \neq 0$$

therefore the general solution can be written as $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

Objective : Write the general solution in terms of real valued functions.

Recall the principle of superposition for the complex case

If $y(t) = u(t) + iv(t)$ solves $L[y] = 0$, then $u(t)$ and $v(t)$ are solutions of $L[y] = 0$.

In our case $\quad y_1(t) = e^{-t}\cos(\sqrt{3}t) + i e^{-t}\sin(\sqrt{3}t)$

$$y_2(t) = e^{-t}\cos(\sqrt{3}t) - i e^{-t}\sin(\sqrt{3}t)$$

so $e^{-t}\cos(\sqrt{3}t)$ and $e^{-t}\sin(\sqrt{3}t)$ are solutions.

Infact, they form a fundamental set of solutions

Check

$$W\left(e^{-t}\cos(\sqrt{3}t),\ e^{-t}\sin(\sqrt{3}t)\right) = e^{-t}\cos(\sqrt{3}t)\cdot\left(-e^{-t}\sin(\sqrt{3}t) + e^{-t}\cos(\sqrt{3}t)\cdot\sqrt{3}\right)$$
$$- e^{-t}\sin(\sqrt{3}t)\cdot\left(-e^{-t}\cos(\sqrt{3}t) + e^{-t}\sin(\sqrt{3}t)\cdot\sqrt{3}\right)$$

algebra

$$\left[-e^{-2t}\cos(\sqrt{3}t)(\sin(\sqrt{3}t)) + \sqrt{3}e^{-2t}\cos^2(\sqrt{3}t)\right] - \left[-e^{-2t}\sin(\sqrt{3}t)\cos(\sqrt{3}t) - e^{-2t}\cdot\sqrt{3}\sin^2(\sqrt{3}t)\right]$$

$$= \sqrt{3}e^{-2t}\left(\cos^2(\sqrt{3}t) + \sin^2(\sqrt{3}t)\right) = \sqrt{3}e^{-2t} \neq 0$$

therefore $e^{-t}\cos(\sqrt{3}t)$ and $e^{-t}\sin(\sqrt{3}t)$ are fundamental solutions.

The general solution is

$$\boxed{y(t) = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t)}$$

Example #2

Solve

$$y'' - 2y' + 5y = 0, \quad y\left(\tfrac{\pi}{2}\right) = 0, \quad y'\left(\tfrac{\pi}{2}\right) = 0$$

Summary of case II

If $b^2 - 4ac < 0$ and $ar^2 + br + c = 0$ has 2 roots $r_{1,2} = \lambda \pm i\mu$ then

$$u(t) = e^{\lambda t}\cos(\mu t) \qquad v(t) = e^{\lambda t}\sin(\mu t)$$

is a fundamental set of real valued solutions with

$$W[u,v](t) = \mu e^{2\lambda t} \neq 0 \text{ provided } \mu \neq 0.$$

The general solution is

$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t).$$

Example #2 (Solutions)

Solve $y'' - 2y' + 5y = 0$, $y\left(\frac{\pi}{2}\right) = 0$, $y'\left(\frac{\pi}{2}\right) = 2$

Characteristic polynomial

$$r^2 - 2r + 5 = 0.$$

$$b^2 - 4ac = (-2)^2 - 4.1.5 = 4 - 20 = -16 < 0, \text{ so}$$

$$r = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm i\sqrt{16}}{2} = 1 \pm i \cdot 2 = 1 \pm 2i$$

$u(t) = e^t \cos(2t)$ and $v(t) = e^t \sin(2t)$ are fundamental solutions.

$$y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t).$$

Applying initial conditions

$$y\left(\frac{\pi}{2}\right) = 0 \Rightarrow c_1 e^{\frac{\pi}{2}} \cos(\pi) + c_2 e^{\frac{\pi}{2}} \sin(\pi) = 0$$

$$-c_1 e^{\frac{\pi}{2}} = 0 \Rightarrow c_1 = 0 \Rightarrow y = c_2 e^t \sin(2t)$$

$$y'\left(\frac{\pi}{2}\right) = 0$$

$$y'(t) = c_2\left(e^t \sin(2t) + 2\cos(2t) e^t\right)$$

$$y'\left(\frac{\pi}{2}\right) = 0 \Rightarrow c_2\left(e^{\frac{\pi}{2}} \sin(\pi) + 2\cos(\pi) e^{\frac{\pi}{2}}\right) = 2$$

$$-c_2 \cdot 2 e^{\frac{\pi}{2}} = 2 \Rightarrow c_2 = -e^{-\frac{\pi}{2}}$$

$$y(t) = -e^{-\frac{\pi}{2}} \cdot e^t \sin(2t)$$

$$= -e^{t - \frac{\pi}{2}} \cdot \sin(2t).$$

$y(t)$ oscillates with increasing amplitude.