3.4

Repeated roots $(b^2 - 4ac = 0)$

i.e. the characteristic polynomial has 1 repeated root $\boxed{r = \frac{-b}{2a}}$

Observe that if $r = \frac{-b}{2a}$, then $2ar = -b \Rightarrow 2ar + b = 0$.

FACT

$y(t) = te^{\frac{-b}{2a}t}$ is a solution to $ay'' + by' + cy = 0$.

for simplicity, let $\bar{r} = \frac{-b}{2a}$ so that $y(t) = te^{\bar{r}t}$

$y'(t) = t\bar{r}e^{\bar{r}t} + e^{\bar{r}t} = e^{\bar{r}t}(\bar{r}t + 1) = \bar{r}te^{\bar{r}t} + e^{\bar{r}t}$

$y''(t) = \bar{r}e^{\bar{r}t}(\bar{r}t + 1) + e^{\bar{r}t}\cdot\bar{r} = \bar{r}^2te^{\bar{r}t} + 2\bar{r}e^{\bar{r}t}$

Plug into $ay'' + by' + cy = 0$

$a(\bar{r}^2te^{\bar{r}t} + 2\bar{r}e^{\bar{r}t}) + b(\bar{r}te^{\bar{r}t} + e^{\bar{r}t}) + c(te^{\bar{r}t})$

$= te^{\bar{r}t}\underbrace{(a\bar{r}^2 + b\bar{r} + c)}_{0} + \underbrace{(2a\bar{r} + b)}_{0}e^{\bar{r}t} = 0$

$\{e^{\bar{r}t}, te^{\bar{r}t}\}$ are fundamental solutions.

Check

$W(e^{\bar{r}t}, te^{\bar{r}t}) = e^{\bar{r}t}\cdot\frac{d}{dt}(te^{\bar{r}t}) - \frac{d}{dt}(e^{\bar{r}t})(te^{\bar{r}t})$

$= e^{\bar{r}t}(t\cdot\bar{r}e^{\bar{r}t} + e^{\bar{r}t}\cdot 1) - (\bar{r}e^{\bar{r}t})(te^{\bar{r}t})$

$= \bar{r}te^{\bar{r}t}e^{\bar{r}t} + e^{\bar{r}t}\cdot e^{\bar{r}t} - \bar{r}te^{\bar{r}t}\cdot e^{\bar{r}t}$

$= e^{2\bar{r}t} \neq 0$ for all t

The general solution is $\boxed{y = C_1e^{\bar{r}t} + C_2te^{\bar{r}t}}$

Example

Solve $4y'' + 4y' + y = 0$, $y(0) = 1$, $y'(0) = 2$.

Characteristic polynomial

$4r^2 + 4r + 1 = 0$

$r = \frac{-4 \pm \sqrt{4^2 - (4.4.1)}}{2\cdot 4} = \frac{-4}{8} = -\frac{1}{2}$

The general solution is $\boxed{y(t) = c_1 e^{-\frac{t}{2}} + c_2 t e^{-\frac{t}{2}}}$

$$y(0) = 1 \Rightarrow \quad c_1 + 0 = 1 \quad \Rightarrow \quad c_1 = 1. \Rightarrow y(t) = e^{-\frac{t}{2}} + c_2 t e^{-\frac{t}{2}}$$

$$y'(t) = -\frac{1}{2} e^{-\frac{t}{2}} + c_2 \left(t \cdot \left(-\frac{1}{2} e^{-\frac{t}{2}}\right) + e^{-\frac{t}{2}} \cdot 1 \right)$$

$$y'(0) = 2 \Rightarrow \quad -\frac{1}{2} + c_2 = 2 \Rightarrow c_2 = \frac{5}{2}$$

$$y(t) = e^{-\frac{t}{2}} + \frac{5}{2} t e^{-\frac{t}{2}}$$