Non-homogeneous equations (PART A)

Method of undetermined coefficients

Solve a second order ODE of the form

$$\mathcal{L}[y] = ay'' + by' + cy = g(t)$$

Recap from last time...

- General solution of $\mathcal{L}[y] = g(t)$ can be written in the form

$$\boxed{y(t) = y_c(t) + y_p(t)}$$

where

$y_c(t)$(complimentary solution) is the solution to $\mathcal{L}[y] = 0$

and

$y_p(t)$(a particular solution) is any solution to $\mathcal{L}[y] = g(t)$

Method of undetermined coefficients

- This means finding the general solution to the non-homogeneous problem consists of **2 steps:**

  STEP 1 Solve the homogeneous problem to find $y_c(t)$

  STEP 2 Find a particular solution to the non-homogeneous problem

- We already know STEP 1 so we focus on how to find a **particular solution**
- The technique depends on **guessing** the form of the solution depending on the form of $g(t)$

Example #1 (polynomial rhs)

Find a **particular solution** to

$$y'' + 3y' + 2y = 3t$$

- A particular solution, $y_p(t)$ has to be a LINEAR function for the sum of the derivatives to be linear.
- So we seek a particular solution of the form $y_p(t) = At + B$.

Solution

Since $y_p(t) = At + B$

$$\boxed{y_p'(t) = A \text{ and } y_p''(t) = 0}$$

and plugging that into the ODE

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= 0 + 3A + 2(At + B) \\ &= 2At + (3A + 2B) \end{aligned}$$

Example #1 ...

$$y_p'' + 3y_p' + 2y_p = 0 + 3A + 2(At + B)$$
$$= 2At + (3A + 2B) = 3t$$

Comparing coefficients of t we need

$$2A = 3 \implies A = \frac{3}{2}$$

and the constant term

$$(3A + 2B) = 0 \implies 3A = -2B \implies B = -\frac{3}{2}A = -\frac{9}{4}$$

Therefore a particular solution is $y_p(t) = \frac{3}{2}t - \frac{9}{4}$

Example #1 ...

The general solution

- Recall that $y(t) = y_c(t) + y_p(t)$, (we have just found $y_p(t)$)
- To find $y_c(t)$ we solve the homogeneous problem

$$\boxed{y'' + 3y' + 2y = 0}$$

The characteristic equation is $r^2 + 3r + 2 = 0$ and solving

$$r^2 + 3r + 2 = (r+1)(r+2) = 0 \Longrightarrow r = -1, -2$$

so

$$y_c(t) = c_1 e^{-t} + c_2 e^{-2t}$$

and finally, the general solution is

$$\boxed{y(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{3}{2}t - \frac{9}{4}}$$

Example #2 (exponential rhs)

Find a **particular solution** to

$$y'' + 3y' + 2y = 10e^{3t}$$

- $y_p(t)$ has to be an EXPONENTIAL function for the sum of the derivatives to be $10e^{3t}$
- So we seek a particular solution of the form $y_p(t) = Ae^{3t}$.

Solution

Since $y_p(t) = Ae^{3t}$

$$\boxed{y_p'(t) = 3Ae^{3t} \text{ and } y_p''(t) = 9Ae^{3t}}$$

and plugging that into the ODE

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= 9Ae^{3t} + 3(3Ae^{3t}) + 2(Ae^{3t}) \\ &= 20Ae^{3t} \end{aligned}$$

Example #2 ...

$$y_p'' + 3y_p' + 2y_p = 9Ae^{3t} + 3(3Ae^{3t}) + 2(Ae^{3t})$$
$$= 20Ae^{3t} = 10e^{3t}$$

Comparing coefficients of e^{3t} we need

$$20A = 10 \Longrightarrow A = \frac{10}{20} = \frac{1}{2}$$

Therefore the particular solution, $y_p(t) = \frac{1}{2}e^{3t}$

Example #3 (trigonometric rhs)

Find a **particular solution** to

$$y'' + 3y' + 2y = \sin(t)$$

- $y_p(t)$ has to be a trigonometric function for the sum of the derivatives to be sines and cosines
- So we seek a particular solution of the form $y_p(t) = A\sin(t) + B\cos(t)$.

Solution

Since $y_p(t) = A\sin(t) + B\cos(t)$

$$\boxed{y_p'(t) = A\cos(t) - B\sin(t) \text{ and } y_p''(t) = -A\sin(t) - B\cos(t)}$$

and plug that into the ODE to get

$$\begin{aligned}
&(-A\sin(t) - B\cos(t)) + 3(A\cos(t) - B\sin(t)) + 2(A\sin(t) + B\cos(t)) \\
&= (-A - 3B + 2A)\sin(t) + (-B + 3A + 2B)\cos(t) \\
&= (A - 3B)\sin(t) + (3A + B)\cos(t)
\end{aligned}$$

Example #3...

Solution

$$y'' + 3y' + 2y = (A - 3B)\sin(t) + (3A + B)\cos(t) = \sin(t)$$

Comparing coefficients of $\sin(t)$ and $\cos(t)$

$$\begin{aligned} A - 3B &= 1 \quad (1) \\ 3A + B &= 0 \quad (2) \end{aligned}$$

From equation (2), $B = -3A$, then plug into (1) to get $A - 3(-3A) = 1$ so that

$$10A = 1 \implies A = \frac{1}{10}, B = -\frac{3}{10}$$

We conclude that

$$\boxed{y_p(t) = \frac{1}{10}\sin(t) - \frac{3}{10}\cos(t)}$$

Solve the following

Practice problem

Find a general solution of

$$y'' - 3y' - 4y = 2\sin(t)$$

Notes

- First find the complimentary solution, i.e the solution to $y'' - 3y' - 4y = 0$
- Since our rhs is trigonometric, we seek $y_p(t) = A\sin(t) + B\cos(t)$

Your solution should be

$$y(t) = c_1e^{-t} + c_2e^{4t} + \left(-\frac{5}{17}\sin(t) + \frac{3}{17}\cos(t)\right)$$