Non-homogeneous equations (PART B)
Method of undetermined coefficients

Solve a second order ODE of the form

\[ L[y] = ay'' + by' + cy = g(t) \]
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- We consider various combinations of polynomials, exponential and trigonometric functions
Example #4

Find a particular solution to

\[ y'' + 3y' + 2y = 4e^{-t} \cos(2t) \]
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- Observe that \( g(t) \) is a product of \textit{exponential} and \textit{trigonometric} functions.
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Find a particular solution to

\[ y'' + 3y' + 2y = 4e^{-t} \cos(2t) \]

- Observe that \( g(t) \) is a product of exponential and trigonometric functions
- So we seek a particular solution of the form

\[ y_p(t) = Ae^{-t} \cos(2t) + Be^{-t} \sin(2t) \]
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- So we seek a particular solution of the form

\[
y_p(t) = Ae^{-t} \cos(2t) + Be^{-t} \sin(2t)
\]

Solution

\[
y_p'(t) = A[-e^{-t} \cos(2t) + e^{-t}(-2 \sin(2t))] + B[-e^{-t} \sin(2t) + e^{-t}(2 \cos(2t))]
\]

\[
= -Ae^{-t} \cos(2t) - 2Ae^{-t} \sin(2t) - Be^{-t} \sin(2t) + 2Be^{-t} \cos(2t)
\]

\[
= [2B - A](e^{-t} \cos(2t)) + [-2A - B](e^{-t} \sin(2t))
\]
Example #4...

\[ y_p(t) = Ae^{-t} \cos (2t) + Be^{-t} \sin (2t) \]
\[ y'_p(t) = [2B - A](e^{-t} \cos (2t)) + [-2A - B](e^{-t} \sin (2t)) \]

therefore (careful use of product and chain rules!)

\[ y''_p(t) = [2B - A][-e^{-t} \cos (2t) + e^{-t}(-2 \sin (2t))] \]
\[ + [-2A - B][-e^{-t} \sin (2t) + e^{-t}(2 \cos (2t))] \]
\[ = [-4B - 3A](e^{-t} \cos (2t)) + [-3B + 4A](e^{-t} \sin (2t)) \]
and plugging into the ODE $y_p''(t) + 3y_p'(t) + 2y_p(t)$

$$= ([−4B − 3A] + [3(2B − A)] + 2A)(e^{−t} \cos (2t))$$
$$+ ([−3B + 4A] + [3(−2A − B)] + 2B)(e^{−t} \sin (2t))$$
$$= [2B − 4A](e^{−t} \cos (2t)) + [−4B − 2A](e^{−t} \sin (2t))$$
$$= 4e^{−t} \cos (2t)$$
and plugging into the ODE $y''_p(t) + 3y'_p(t) + 2y_p(t)$

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$$+ ([−3B + 4A] + [3(−2A − B)] + 2B)(e^{−t} \sin (2t))$$
$$= [2B − 4A](e^{−t} \cos (2t)) + [−4B − 2A](e^{−t} \sin (2t))$$
$$= 4e^{−t} \cos (2t)$$

Comparing coefficients of $e^{−t} \cos (2t)$ and $e^{−t} \sin (2t)$

$$2B − 4A = 4$$
$$−4B − 2A = 0$$
Example #4...

Comparing coefficients of $e^{-t}\cos(2t)$ and $e^{-t}\sin(2t)$

\[
\begin{align*}
2B - 4A & = 4 \\
-4B - 2A & = 0
\end{align*}
\] (1) (2)

From equation (2) $-4B = 2A \implies A = -2B$, then plug into (1)

\[
2B - 4(-2B) = 4 \text{ so that } 10B = 4 \implies B = \frac{2}{5}, A = -\frac{4}{5}
\]

We conclude

\[
y_p(t) = -\frac{4}{5}e^{-t}\cos(2t) + \frac{2}{5}e^{-t}\sin(2t)
\]
An example for you to do

Practice problem

\[ y'' + 4y = 5t^2 e^t \]
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Practice problem

\[ y'' + 4y = 5t^2e^t \]

Hints

1. First solve the homogeneous problem \( y'' + 4y = 0 \)
2. The right hand side function \( g(t) \) is a product of a quadratic and an exponential so

\[ y_p(t) = (At^2 + Bt + C)e^t \]
An example for you to do

Practice problem

\[ y'' + 4y = 5t^2 e^t \]

Hints

1. First solve the homogeneous problem \( y'' + 4y = 0 \)
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Your general solution should be

\[ y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \left( t^2 - \frac{4t}{5} - \frac{2}{25} \right) e^t \]
Example #5 (sums)

Solve

\[ y'' + 3y' + 2y = 3t - 10e^{3t} + \sin(t) + 4e^{-t}\cos(2t) \]
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The solution to the homogeneous problem is

\[ y_c(t) = c_1 e^{-t} + c_2 e^{-2t} \]
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- The solution to the homogeneous problem is
  \[ y_c(t) = c_1 e^{-t} + c_2 e^{-2t} \]

- We have found particular solutions for
  \[ \text{Ex } \#1 \quad y'' + 3y' + 2y = 3t \implies y_p^1(t) = \frac{3}{2} t - \frac{9}{4} \]
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  \text{Ex #1} & \quad y'' + 3y' + 2y = 3t \implies y_p^1(t) = \frac{3}{2}t - \frac{9}{4} \\
  \text{Ex #2} & \quad y'' + 3y' + 2y = 10e^{3t} \implies y_p^2(t) = \frac{1}{2}e^{3t}
  \end{align*}
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  - Ex #2 \( y'' + 3y' + 2y = 10e^{3t} \implies y_p(t) = \frac{1}{2}e^{3t} \)
  - Ex #3 \( y'' + 3y' + 2y = \sin(t) \implies y_p(t) = \frac{1}{10} \sin(t) - \frac{3}{10} \cos(t) \)
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Solve

\[ y'' + 3y' + 2y = 3t - 10e^{3t} + \sin(t) + 4e^{-t}\cos(2t) \]

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  \begin{align*}
  \text{Ex #1} & \quad y'' + 3y' + 2y = 3t \quad \implies \quad y_1^p(t) = \frac{3}{2}t - \frac{9}{4} \\
  \text{Ex #2} & \quad y'' + 3y' + 2y = 10e^{3t} \quad \implies \quad y_2^p(t) = \frac{1}{2}e^{3t} \\
  \text{Ex #3} & \quad y'' + 3y' + 2y = \sin(t) \quad \implies \quad y_3^p(t) = \frac{1}{10}\sin(t) - \frac{3}{10}\cos(t) \\
  \text{Ex #4} & \quad y'' + 3y' + 2y = 4e^{-t}\sin(t) \quad \implies \quad y_4^p(t) = -\frac{4}{5}e^{-t}\cos(2t) - \frac{2}{5}\sin(2t)
  \end{align*}
Example #5 (sums)

Solve

$$y'' + 3y' + 2y = 3t - 10e^{3t} + \sin(t) + 4e^{-t} \cos(2t)$$

1. The solution to the homogeneous problem is

$$y_c(t) = c_1 e^{-t} + c_2 e^{-2t}$$

2. We have found particular solutions for

   Ex #1  $$y'' + 3y' + 2y = 3t \implies y_p^1(t) = \frac{3}{2} t - \frac{9}{4}$$
   Ex #2  $$y'' + 3y' + 2y = 10e^{3t} \implies y_p^2(t) = \frac{1}{2} e^{3t}$$
   Ex #3  $$y'' + 3y' + 2y = \sin(t) \implies y_p^3(t) = \frac{1}{10} \sin(t) - \frac{3}{10} \cos(t)$$
   Ex #4  $$y'' + 3y' + 2y = 4e^{-t} \sin(t) \implies y_p^4(t) = -\frac{4}{5} e^{-t} \cos(2t) - \frac{2}{5} \sin(2t)$$

Therefore the general solution to our problem is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + y_p^1(t) - y_p^2(t) + y_p^3(t) + y_p^4(t)$$