Non-homogeneous equations (PART B)

Method of undetermined coefficients

Solve a second order ODE of the form

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- We consider various combinations of polynomials, exponential and trigonometric functions

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Solution

$$\begin{aligned} y_p'(t) &= A\big[-e^{-t}\cos(2t) + e^{-t}(-2\sin(2t))\big] + B\big[-e^{-t}\sin(2t) + e^{-t}(2\cos(2t))\big] \\ &= -Ae^{-t}\cos(2t) - 2Ae^{-t}\sin(2t) - Be^{-t}\sin(2t) + 2Be^{-t}\cos(2t) \\ &= [2B - A](e^{-t}\cos(2t)) + [-2A - B](e^{-t}\sin(2t)) \end{aligned}$$

Example #4...

$$y_p(t) = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)$$
$$y_p'(t) = [2B - A](e^{-t}\cos(2t)) + [-2A - B](e^{-t}\sin(2t))$$

therefore (careful use of product and chain rules!)

$$\begin{aligned} y_p''(t) &= [2B - A]\{-e^{-t}\cos(2t) + e^{-t}(-2\sin(2t))\} \\ &\quad + [-2A - B]\{-e^{-t}\sin(2t) + e^{-t}(2\cos(2t))\} \\ &= [-4B - 3A](e^{-t}\cos(2t)) + [-3B + 4A](e^{-t}\sin(2t)) \end{aligned}$$

Example #4...

and plugging into the ODE $y_p''(t) + 3y_p'(t) + 2y_p(t)$

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\begin{aligned}
&= \big([-4B - 3A] + [3(2B - A)] + 2A\big)(e^{-t}\cos(2t)) \\
&+ \big([-3B + 4A] + [3(-2A - B)] + 2B\big)(e^{-t}\sin(2t)) \\
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Comparing coefficients of $e^{-t}\cos(2t)$ and $e^{-t}\sin(2t)$

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\begin{aligned}
2B - 4A &= 4 \\
-4B - 2A &= 0
\end{aligned}
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Comparing coefficients of $e^{-t}\cos(2t)$ and $e^{-t}\sin(2t)$

$$\begin{aligned} 2B - 4A &= 4 \qquad (1) \\ -4B - 2A &= 0 \qquad (2) \end{aligned}$$

From equation (2) $-4B = 2A \Longrightarrow A = -2B$, then plug into (1)

$$2B - 4(-2B) = 4 \text{ so that } 10B = 4 \Longrightarrow B = \frac{2}{5}, A = -\frac{4}{5}$$

We conclude

$$\boxed{y_p(t) = -\frac{4}{5}e^{-t}\cos(2t) + \frac{2}{5}e^{-t}\sin(2t)}$$

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Practice problem

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Hints

1. First solve the homogeneous problem $y'' + 4y = 0$
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$$y_p(t) = (At^2 + Bt + C)e^t$$

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Your general solution should be

$$\boxed{y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \left(t^2 - \frac{4t}{5} - \frac{2}{25}\right)e^t}$$

Example #5 (sums)

Solve

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therefore the general solution to our problem is

$$\boxed{y(t) = c_1 e^{-t} + c_2 e^{-2t} + y_p^1(t) - y_p^2(t) + y_p^3(t) + y_p^4(t)}$$