Non-homogeneous equations (PART C)

Method of undetermined coefficients

Solve a second order ODE of the form

$$\mathcal{L}[y] = ay'' + by' + cy = g(t)$$

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- We consider some exceptions to the rules covered in PARTS A and B

Example #6a

Find a **particular solution** to

$$y'' + y' = 5$$

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- $g(t)$ is constant so following our logic from previous examples
- Let's try particular solution of the form

$$\boxed{y_p(t) = A}$$

First compute

$$y_p'(t) = 0 \text{ and } y_p''(t) = 0,$$

then plugging into ODE yields

$$y_p'' + y_p' = 0 \neq 5$$

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$$y_p'' + y_p' = 0 \neq 5$$

$y_p(t) = A$ is already a solution to the homogeneous problem!

Example #6b

Find a **particular solution** to

$$y'' - 6y' + 9y = e^{3t}$$

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- $g(t)$ is exponential so let's try particular solution of the form

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$$y'' - 6y' + 9y = e^{3t}$$

- $g(t)$ is exponential so let's try particular solution of the form

$$\boxed{y_p(t) = Ae^{3t}}$$

First compute

$$y_p'(t) = 3Ae^{3t} \text{ and } y_p''(t) = 9Ae^{3t}$$

then plugging into ODE yields,

$$y_p'' - 6y_p' + 9y_p = 9Ae^{3t} - 6(3Ae^{3t}) + 9(Ae^{3t}) = 0 \neq e^{3t}$$

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$y_p(t) = e^{3t}$ is already a solution to the homogeneous problem, in fact $y_p(t) = te^{3t}$ is also a solution. Check the characteristic polynomial.

The moral of the 2 examples . . .

- Before settling on the form of the particular solution, $y_p(t)$ we need to find the complimentary solution and make sure that our choice of $y_p(t)$ is not a solution to the homogeneous problem.

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- Before settling on the form of the particular solution, $y_p(t)$ we need to find the complimentary solution and make sure that our choice of $y_p(t)$ is not a solution to the homogeneous problem.
- The fix...

> Recall the repeated roots case of the characteristic polynomial. We multiplied the only exponential solution by a factor of t. We will do the same here.

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The characteristic polynomial is

$$r^2 + r = 0 \iff r(r+1) = 0 \implies r = 0, -1$$

so

$$y_c(t) = c_1 e^0 + c_2 e^{-t} = c_1 + c_2 e^{-t}$$

Example #6a

Find a **particular solution** to

$$y'' + y' = 5$$

- First solve the homogeneous problem

The characteristic polynomial is

$$r^2 + r = 0 \Longleftrightarrow r(r+1) = 0 \Longrightarrow r = 0, -1$$

so

$$y_c(t) = c_1 e^0 + c_2 e^{-t} = c_1 + c_2 e^{-t}$$

- Since any constant c_1 is a solution, we cannot choose $y_p(t) = A$, instead we choose

$$\boxed{y_p(t) = At}$$

Example #6a

$y_p(t) = At$, so

$$y_p'(t) = A \text{ and } y_p''(t) = 0$$

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then plugging into ODE yields,

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$y_p(t) = At$, so

$$y_p'(t) = A \text{ and } y_p''(t) = 0$$

then plugging into ODE yields,

$$y_p'' + y_p' = 0 + A = 5 \Longrightarrow A = 5$$

So $y_p(t) = 5t$ and the general solution is

$$\boxed{y(t) = = c_1 + c_2 e^{-t} + 5t}$$

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$$y'' - 6y' + 9y = e^{3t}$$

- First, solve the homogeneous problem

The characteristic polynomial is

$$r^2 - 6r + 9 = 0 \iff (r-3)^2 = 0 \implies r = 3, \text{ twice.}$$

so

$$y_c(t) = c_1 e^{3t} + c_2 t e^{3t}$$

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so

$$y_c(t) = c_1 e^{3t} + c_2 t e^{3t}$$

- Since any constant both e^{3t} and te^{3t} are solutions, we cannot choose $y_p(t) = Ae^{3t}$ or $y_p(t) = Ate^{3t}$, instead we choose

$$\boxed{y_p(t) = At^2 e^{3t}}$$

Example #6b

- $y_p(t) = At^2e^{3t}$ so

$$\begin{aligned}
y_p'(t) &= A(3t^2e^{3t} + 2te^{3t}) = 3At^2e^{3t} + 2Ate^{3t} \\
y_p''(t) &= 6Ate^{3t} + 9At^2e^{3t} + 2Ae^{3t} + 6Ate^{3t} \\
&= 2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t}
\end{aligned}$$

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$$= 2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t}$$

Plug into ODE:

$$y_p'' - 6y_p' + 9y_p$$
$$= \{2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t}\} - 6\{3At^2e^{3t} + 2Ate^{3t}\} + 9At^2e^{3t}$$
$$= 2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t} - 18At^2e^{3t} - 12Ate^{3t} + 9At^2e^{3t}$$
$$= 2Ae^{3t} = e^{3t} \Longrightarrow 2A = 1 \Longrightarrow A = \frac{1}{2}$$

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- $y_p(t) = At^2e^{3t}$ so

$$y_p'(t) = A(3t^2e^{3t} + 2te^{3t}) = 3At^2e^{3t} + 2Ate^{3t}$$
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$$= 2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t}$$

Plug into ODE:

$$y_p'' - 6y_p' + 9y_p$$
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$$= 2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t} - 18At^2e^{3t} - 12Ate^{3t} + 9At^2e^{3t}$$
$$= 2Ae^{3t} = e^{3t} \implies 2A = 1 \implies A = \frac{1}{2}$$

- Therefore $y_p(t) = \dfrac{1}{2}t^2e^{3t}$

Example #7a

Find the form of a **particular solution** to

$$y'' + 3y' = 2t^4$$

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- First, solve the homogeneous problem

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$$y_c(t) = c_1 e^0 + c_2 e^{-3t} = c_1 + c_2 e^{-3t}$$

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- Any constant is a solution to the homogeneous problem and since that is contained in

$$y_p(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4$$

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- Any constant is a solution to the homogeneous problem and since that is contained in

$$y_p(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4$$

- We choose instead

$$y_p(t) = t(A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4)$$

Example #7b

Find the form of a **particular solution** to

$$y'' + 3y' = t^2 e^{-3t}$$

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The characteristic polynomial is

$$r^2 + 3r = 0 \iff r(r+3) = 0 \implies r = 0, -3$$

so

$$y_c(t) = c_1 e^0 + c_2 e^{-3t} = c_1 + c_2 e^{-3t}$$

- $c_2 e^{-3t}$ is a solution to the homogeneous problem and since that is contained in

$$y_p(t) = (B_0 + B_1 t + B_2 t^2) e^{-3t}$$

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- we choose instead

$$y_p(t) = t(B_0 + B_1 t + B_2 t^2) e^{-3t}$$

Example #7c

Find the form of a **particular solution** to

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$$y_c(t) = c_1 e^0 + c_2 e^{-3t} = c_1 + c_2 e^{-3t}$$

Example #7c

Find the form of a **particular solution** to

$$y'' + 3y' = \sin(3t)$$

- First, solve the homogeneous problem

The characteristic polynomial is

$$r^2 + 3r = 0 \iff r(r+3) = 0 \implies r = 0, -3$$

so

$$y_c(t) = c_1 e^0 + c_2 e^{-3t} = c_1 + c_2 e^{-3t}$$

- The particular solution looks nothing like the rhs so

$$y_p(t) = C_1 \sin(3t) + C_2 \cos(3t)$$

Example #7c

Find the form of a **particular solution** to

$$y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin(3t)$$

- We simply combine our particular solutions from Examples $7a - c$

$$y(t) = t(A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4)$$
$$+t(B_0 + B_1 t + B_2 t^2)e^{-3t} + C_1 \sin(3t) + C_2 \cos(3t)$$

Summary

Superposition

We can find a particuplar solution, $y_p(t)$

$$ay'' + by' + cy = g_1(t) + g_2(t) + \cdots + g_n(t)$$

by splitting up the problem into smaller subproblems:

$$\begin{aligned}
ay'' + by' + cy &= g_1(t) \text{ with solution } y_p^1(t) \\
ay'' + by' + cy &= g_2(t) \text{ with solution } y_p^2(t) \\
ay'' + by' + cy &= g_3(t) \text{ with solution } y_p^3(t) \\
&\vdots \\
ay'' + by' + cy &= g_n(t) \text{ with solution } y_p^n(t)
\end{aligned}$$

so that $y_p(t) = y_p^1(t) + y_p^2(t) + \cdots + y_p^n(t)$

Summary

Particular solution of $ay'' + by' + cy = g_i(t)$

if $g_i(t)$ is:	$y_p(t)$ is:
$P_n(t)$	$t^s(a_0 + a_1 t + \cdots a_n t^n)$
$P_n(t)e^{\alpha t}$	$t^s e^{\alpha t}(a_0 + a_1 t + \cdots + a_n t^n)$
$P_n(t)e^{\alpha t}\sin(\mu t)$ or $P_n(t)e^{\alpha t}\cos(\mu t)$	$t^s[(a_0 + a_1 t + \cdots + a_n t^n)e^{\alpha t}\sin(\mu t)$ $+ (b_0 + b_1 t + \cdots + b_n t^n)]e^{\alpha t}\cos(\mu t)$

Summary

Particular solution of $ay'' + by' + cy = g_i(t)$

if $g_i(t)$ is:	$y_p(t)$ is:
$P_n(t)$	$t^s(a_0 + a_1 t + \cdots a_n t^n)$
$P_n(t)e^{\alpha t}$	$t^s e^{\alpha t}(a_0 + a_1 t + \cdots + a_n t^n)$
$P_n(t)e^{\alpha t}\sin(\mu t)$ or $P_n(t)e^{\alpha t}\cos(\mu t)$	$t^s[(a_0 + a_1 t + \cdots + a_n t^n)e^{\alpha t}\sin(\mu t)$ $+ (b_0 + b_1 t + \cdots + b_n t^n)]e^{\alpha t}\cos(\mu t)$

Note:

- $s = 0, 1$ or 2 ensures that $y_p(t)$ is not a solution to the homogeneous problem.