Non-homogeneous equations (PART C)

Method of undetermined coefficients

Solve a second order ODE of the form

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• We consider some exceptions to the rules covered in PARTS A and B

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First compute

$$y'_p(t) = 0$$
 and $y''_p(t) = 0$,

then plugging into ODE yields

$$y_p'' + y_p' = 0 \neq 5$$

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 $y_{\rho}(t) = A$ is already a solution to the homogeneous problem!

$$y''-6y'+9y=e^{3t}$$

$$y'' - 6y' + 9y = e^{3t}$$

• g(t) is exponential so let's try particular solution of the form

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• g(t) is exponential so let's try particular solution of the form

$$y_p(t) = Ae^{3t}$$

First compute

$$y'_p(t) = 3Ae^{3t}$$
 and $y''_p(t) = 9Ae^{3t}$

then plugging into ODE yields,

$$y_{p}^{\prime\prime}-6y_{p}^{\prime}+9y_{p}=9Ae^{3t}-6(3Ae^{3t})+9(Ae^{3t})=0
eq e^{3t}$$

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$$y'_p(t) = 3Ae^{3t}$$
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then plugging into ODE yields,

$$y_p'' - 6y_p' + 9y_p = 9Ae^{3t} - 6(3Ae^{3t}) + 9(Ae^{3t}) = 0 \neq e^{3t}$$

 $y_p(t) = e^{3t}$ is already a solution to the homogeneous problem, in fact $y_p(t) = te^{3t}$ is also a solution. Check the characteristic polynomial.

The moral of the 2 examples . . .

• Before settling on the form of the particular solution, $y_p(t)$ we need to find the complimentary solution and make sure that our choice of $y_p(t)$ is not a solution to the homogeneous problem.

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- Before settling on the form of the particular solution, $y_p(t)$ we need to find the complimentary solution and make sure that our choice of $y_p(t)$ is not a solution to the homogeneous problem.
- The fix...

Recall the repeated roots case of the characteristic polynomial. We multiplied the only exponential solution by a factor of t. We will do the same here.

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The characteristic polynomial is
$$r^2 + r = 0 \iff r(r+1) = 0 \implies r = 0, -1$$
 so $y_c(t) = c_1 e^0 + c_2 e^{-t} = c_1 + c_2 e^{-t}$

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Since any constant c₁ is a solution, we cannot choose y_p(t) = A, instead we choose

$$y_p(t) = At$$

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then plugging into ODE yields,

$$y_p'' + y_p' = 0 + A = 5 \Longrightarrow A = 5$$

= 0

$$y_p(t) = At$$
, so $y_p'(t) = A$ and $y_p''(t) = 0$

then plugging into ODE yields,

$$y_p'' + y_p' = 0 + A = 5 \Longrightarrow A = 5$$

So $y_p(t) = 5t$ and the general solution is

$$y(t) = = c_1 + c_2 e^{-t} + 5t$$

$$y'' - 6y' + 9y = e^{3t}$$

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• First, solve the homogeneous problem

The characteristic polynomial is
$$r^2 - 6r + 9 = 0 \iff (r-3)^2 = 0 \implies r = 3$$
, twice.
so $y_c(t) = c_1 e^{3t} + c_2 t e^{3t}$

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• Since any constant both e^{3t} and te^{3t} are solutions, we cannot choose $y_p(t) = Ae^{3t}$ or $y_p(t) = Ate^{3t}$, instead we choose

$$y_p(t) = At^2 e^{3t}$$

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 so
 $y'_p(t) = A(3t^2 e^{3t} + 2te^{3t}) = 3At^2 e^{3t} + 2Ate^{3t}$
 $y''_p(t) = 6Ate^{3t} + 9At^2 e^{3t} + 2Ae^{3t} + 6Ate^{3t}$
 $= 2Ae^{3t} + 12Ate^{3t} + 9At^2 e^{3t}$

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Plug into ODE:

$$y_{p}^{\prime\prime} - 6y_{p}^{\prime} + 9y_{p}$$

$$= \left\{ 2Ae^{3t} + 12Ate^{3t} + 9At^{2}e^{3t} \right\} - 6\left\{ 3At^{2}e^{3t} + 2Ate^{3t} \right\} + 9At^{2}e^{3t}$$

$$= 2Ae^{3t} + 12Ate^{3t} + 9At^{2}e^{3t} - 18At^{2}e^{3t} - 12Ate^{3t} + 9At^{2}e^{3t}$$

$$= 2Ae^{3t} = e^{3t} \Longrightarrow 2A = 1 \Longrightarrow A = \frac{1}{2}$$

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 so
 $y'_p(t) = A(3t^2 e^{3t} + 2te^{3t}) = 3At^2 e^{3t} + 2Ate^{3t}$
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 $= 2Ae^{3t} + 12Ate^{3t} + 9At^2 e^{3t}$

Plug into ODE:

$$y_{p}'' - 6y_{p}' + 9y_{p}$$

$$= \{2Ae^{3t} + 12Ate^{3t} + 9At^{2}e^{3t}\} - 6\{3At^{2}e^{3t} + 2Ate^{3t}\} + 9At^{2}e^{3t}$$

$$= 2Ae^{3t} + 12Ate^{3t} + 9At^{2}e^{3t} - 18At^{2}e^{3t} - 12Ate^{3t} + 9At^{2}e^{3t}$$

$$= 2Ae^{3t} = e^{3t} \Longrightarrow 2A = 1 \Longrightarrow A = \frac{1}{2}$$

• Therefore
$$y_p(t) = \frac{1}{2}t^2e^{3t}$$

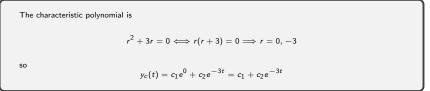
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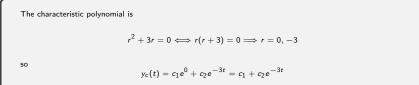
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• Any constant is a solution to the homogeneous problem and since that is contained in

$$y_{\rho}(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4$$

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The characteristic polynomial is
$$r^2 + 3r = 0 \iff r(r+3) = 0 \implies r = 0, -3$$
 so
$$y_c(t) = c_1 e^0 + c_2 e^{-3t} = c_1 + c_2 e^{-3t}$$

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$$y_{\rho}(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4$$

• We choose instead

$$y_p(t) = t(A_0 + A_1t + A_2t^2 + A_3t^3 + A_4t^4)$$

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The characteristic polynomial is $r^2+3r=0 \iff r(r+3)=0 \implies r=0,-3$ so $y_c(t)=c_1e^0+c_2e^{-3t}=c_1+c_2e^{-3t}$

• $c_2 e^{-3t}$ is a solution to the homogeneous problem and since that is contained in

$$y_p(t) = (B_0 + B_1 t + B_2 t^2)e^{-3t}$$

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$$y_c(t) = c_1 e^0 + c_2 e^{-3t} = c_1 + c_2 e^{-3t}$$

• The particular solution looks nothing like the rhs so

$$y_p(t) = C_1 \sin(3t) + C_2 \cos(3t)$$

Find the form of a particular solution to

$$y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin(3t)$$

• We simply combine our particular solutions from Examples 7a - c

$$y(t) = t(A_0 + A_1t + A_2t^2 + A_3t^3 + A_4t^4)$$

+t(B_0 + B_1t + B_2t^2)e^{-3t} + C_1 sin (3t) + C_2 cos (3t)

Summary

Superposition

We can find a particuplar solution, $y_p(t)$

$$ay'' + by' + cy = g_1(t) + g_2(t) + \dots + g_n(t)$$

by splitting up the problem into smaller subproblems:

$$ay'' + by' + cy = g_1(t)$$
 with solution $y_{\rho}^1(t)$
 $ay'' + by' + cy = g_2(t)$ with solution $y_{\rho}^2(t)$
 $ay'' + by' + cy = g_3(t)$ with solution $y_{\rho}^3(t)$

$$ay'' + by' + cy = g_n(t)$$
 with solution $y_p^n(t)$

:

so that $y_p(t) = y_p^1(t) + y_p^2(t) + \dots + y_p^n(t)$

Summary

Particular solution of $ay'' + by' + cy = g_i(t)$

if $g_i(t)$ is:	$y_p(t)$ is:
$P_n(t)$	$t^{s}(a_{0}+a_{1}t+\cdots a_{n}t^{n})$
$P_n(t)e^{\alpha t}$	$t^s e^{\alpha t} (a_0 + a_1 t + \dots + a_n t^n)$
$P_n(t)e^{\alpha t}\sin(\mu t)$ or	$t^{s}[(a_{0}+a_{1}t+\cdots+a_{n}t^{n})e^{\alpha t}\sin(\mu t)$
$P_n(t)e^{lpha t}\cos(\mu t)$	$+(b_0+b_1t+\cdots+b_nt^n)]e^{lpha t}\cos(\mu t)$

Summary

Particular	solution	of	ay"	+	by'	+	су		gi(t)	
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D(1+)	+ ^s $(2 + 2 + 1) + 2 + n)$
	$\frac{t^s(a_0+a_1t+\cdots a_nt^n)}{t^s e^{\alpha t}(a_0+a_1t+\cdots +a_nt^n)}$
$P_n(t)e^{\alpha t}\sin(\mu t)$ or	$t^{s}[(a_{0}+a_{1}t+\cdots+a_{n}t^{n})e^{\alpha t}\sin(\mu t)]$
$P_n(t)e^{\alpha t}\cos(\mu t)$	$+ (b_0 + b_1t + \cdots + b_nt^n)]e^{\alpha t}\cos(\mu t)$

Note:

s = 0, 1 or 2 ensures that y_p(t) is not a solution to the homogeneous problem.