Variation of Parameters

Solve a second order ODE of the form

$$\mathcal{L}[y] = ay'' + by' + cy = g(t)$$

- We consider cases not covered by the **method of undetermined coefficients**

Example #1

Find a **particular solution** to

$$y'' + y' = \tan(t)$$

$g(t)$ is not a polynomial, exponential or sine/cosine function.

- **STEP 1:** Solve the homogeneous problem

The characteristic polynomial is

$$r^2 + 1 = 0 \Longrightarrow r = \pm i$$

so

$$y_c(t) = c_1 e^{0t} \sin(t) + c_2 e^{0t} \cos(t) = c_1 \sin(t) + c_2 \cos(t)$$

Example #1

- **STEP 2:** Replace c_1 and c_2 in $y_c(t)$ by $u_1(t)$ and $u_2(t)$ so that

$$\boxed{y_p(t) = u_1(t)\sin(t) + u_2(t)\cos(t)}$$

solves the **non-homogeneous** problem

- To find $u_1(t)$ and $u_2(t)$ we substitute $y_p(t)$ into the ODE

Key Observation

If we plug in $y_p(t)$ into the ODE, we only have one equation BUT we have 2 unknowns. We will impose another condition that will simplify the calculation so as to make the system solvable.

Example #1

- $y_p(t) = u_1(t)\sin(t) + u_2(t)\cos(t)$ thefore:

$$y_p'(t) = u_1(t)\cos(t) + u_1'(t)\sin(t) + u_2(t)(-\sin(t)) + u_2'(t)\cos(t)$$

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$$\begin{aligned} y_p'(t) &= u_1(t)\cos(t) + u_1'(t)\sin(t) + u_2(t)(-\sin(t)) + u_2'(t)\cos(t) \\ &= u_1(t)\cos(t) - u_2(t)\sin(t) + u_1'(t)\sin(t) + u_2'(t)\cos(t) \end{aligned}$$

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&= \boxed{u_1(t)\cos(t) - u_2(t)\sin(t)} + \underbrace{u_1'(t)\sin(t) + u_2'(t)\cos(t)}_{\text{we will assume this equals zero}}
\end{aligned}
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Condition 1

$$u_1'(t)\sin(t) + u_2'(t)\cos(t) = 0$$

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- $y_p''(t) = u_1'(t)\cos(t) - u_1(t)\sin(t) - \{u_2'(t)\sin(t) + u_2(t)\cos(t)\}$

Example #1

- Plugging into the ODE

$$y_p'' + y_p = \left[u_1'(t)\cos(t) - u_1(t)\sin(t) - u_2'(t)\sin(t) - u_2(t)\cos(t)\right] + \left[u_1(t)\sin(t) + u_2(t)\cos(t)\right]$$

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- Now we have 2 equations and 2 unknowns!

$$\begin{aligned} u_1'(t)\sin(t) + u_2'(t)\cos(t) &= 0 \\ u_1'(t)\cos(t) - u_2'(t)\sin(t) &= tan(t) \end{aligned}$$

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u_1'(t)\sin(t) + u_2'(t)\cos(t) &= 0 \qquad (1)\\
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- From (1)

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$$u_1'(t)\cos(t) - \big(-u_1'(t)\tan(t)\big)\sin(t) = \tan(t)$$
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$$u_1'(t)\cos(t) - \big(-u_1'(t)\tan(t)\big)\sin(t) = \tan(t)$$

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- Therefore $u_1'(t) = \tan(t)\cos(t) = \frac{\sin(t)}{\cos(t)} \cdot \cos(t) = \sin(t)$.

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- Therefore $u_1'(t) = \tan(t)\cos(t) = \frac{\sin(t)}{\cos(t)} \cdot \cos(t) = \sin(t)$.
- and $u_1(t) = \int \sin(t)\,dt = -\cos(t)$

Example #1

- Recalling that

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- $y_p(t) = (-\cos(t))\sin(t) + (-\ln|\sec(t)+\tan(t)| + \sin(t))\cos(t)$

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- $y_p(t) = (-\cos(t))\sin(t) + (-\ln|\sec(t)+\tan(t)| + \sin(t))\cos(t)$
- The general solution is

$$\begin{aligned} y(t) &= c_1\sin(t) + c_2\cos(t) \\ &+ \big(-\cos(t)\sin(t) + (-\ln|\sec(t)+\tan(t)| + \sin(t))\cos(t)\big) \end{aligned}$$

General second order problem

The **variation of parameters** technique works for nonconstant coefficients case:

$$y'' + p(t)y' + q(t)y = g(t)$$

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  and $\{y_1, y_2\}$ form a fundamental solution set.

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$$\begin{aligned} y_p'(t) &= u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t) \\ &= [u_1(t)y_1'(t) + u_2(t)y_2'(t)] \end{aligned}$$

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- We want $y_p(t)$ to solve the ODE.

$$\begin{aligned} y_p'(t) &= u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t) \\ &= [u_1(t)y_1'(t) + u_2(t)y_2'(t)] \underbrace{+u_1'(t)y_1(t) + u_2'(t)y_2(t)}_{\text{we will assume this equals zero}} \end{aligned}$$

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$$\boxed{y_p''(t) = u_1(t)y_1''(t) + u_1'(t)y_1'(t) + u_2(t)y_2''(t) + u_2'(t)y_2'(t)}$$

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- Plugging into ODE yeilds

$$\left[u_1(t)y_1''(t) + u_1'(t)y_1'(t) + u_2(t)y_2''(t) + u_2'(t)y_2'(t)\right]$$

$$+p(t)\left[u_1(t)y_1'(t) + u_2(t)y_2'(t)\right] + q(t)\left[u_1(t)y_1(t) + u_2(t)y_2(t)\right] = g(t)$$

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- Rearranging terms

$$u_1(t)\Big[\qquad\qquad\qquad\Big] + u_2(t)\Big[\qquad\qquad\qquad\Big]$$
$$+u_2'(t)y_2(t) + u_1'(t)y_1'(t) = g(t)$$

General second order problem

- $y_p'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t)$,therefore

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- Rearranging terms

$$u_1(t)\Big[\underbrace{y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)}_{=0}\Big] + u_2(t)\Big[\underbrace{y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)}_{=0}\Big]$$
$$+u_2'(t)y_2(t) + u_1'(t)y_1'(t) = g(t)$$

General second order problem

- Now we have 2 equations and 2 unknowns!

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\begin{aligned}
u_1'(t)y_1(t) + u_2'(t)y_2(t) &= 0 \\
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\end{aligned}
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- Solving yields:

$$u_2'(t) = \frac{g(t)y_1(t)}{y_2'(t)y_1(t) - y_2(t)y_1'(t)}$$

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- Solving yields:

$$u_2'(t) = \frac{g(t)y_1(t)}{y_2'(t)y_1(t) - y_2(t)y_1'(t)} = \frac{y_1(t)g(t)}{W[y_1, y_2]}$$

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$$u_1'(t) = -\frac{g(t)y_2(t)}{y_2'(t)y_1(t) - y_2(t)y_1'(t)}$$

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$$\begin{aligned} u_1'(t)y_1(t) + u_2'(t)y_2(t) &= 0 \\ u_1'(t)y_1'(t) - u_2'(t)y_2'(t) &= g(t) \end{aligned}$$

- Solving yields:

$$u_2'(t) = \frac{g(t)y_1(t)}{y_2'(t)y_1(t) - y_2(t)y_1'(t)} = \frac{y_1(t)g(t)}{W[y_1, y_2]} \Rightarrow u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1, y_2]}\, dt$$

$$u_1'(t) = -\frac{g(t)y_2(t)}{y_2'(t)y_1(t) - y_2(t)y_1'(t)} = -\frac{y_2(t)g(t)}{W[y_1, y_2]} \Rightarrow u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1, y_2]}\, dt$$

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$$y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t \geq 0$$

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- First, solve the homogeneous problem

The characteristic polynomial is

$$r^2 + 4r + 4 = 0 \iff (r+2)^2 = 0 \implies r = -2, \text{ twice}$$

so

$$y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

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$$u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1, y2]}\,dt \text{ and } u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1, y2]}\,dt$$

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where

$$\boxed{y_1(t) = e^{-2t}, y_2(t) = te^{-2t} \text{ and } g(t) = t^{-2}e^{-2t}}$$

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&= -\int \frac{1}{t}\,dt = -\ln t
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y_p(t) = (-\ln(t))e^{-2t} + -\frac{1}{t}te^{-2t} = -e^{-2t}\ln(t) - e^{-2t}
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- And the general solution

$$\boxed{y(t) = c_1e^{-2t} + c_2te^{-2t} + \left(-e^{-2t}\ln(t) - e^{-2t}\right)}$$