Variation of Parameters
Variation of Parameters : PART B

Solve a second order ODE of the form

\[ \mathcal{L}[y] = y'' + p(t)y' + q(t)y = g(t) \]
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Solve a second order ODE of the form

\[ L[y] = y'' + p(t)y' + q(t)y = g(t) \]

**FACT 1**

If \( p(t), q(t) \) and \( g(t) \) are continuous on an open interval and if \( y_1 \) and \( y_2 \) form a fundamental set of solutions of \( y'' + p(t)y' + q(t)y = 0 \) then the particular solution of the general non-homogeneous problem is

\[
y_p(t) = \left( - \int \frac{y_2(t)g(t)}{W[y_1, y_2](t)} \, dt \right)y_1(t) + \left( \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)} \, dt \right)y_2(t)
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and the general solution is

\[
y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)
\]
Example

Find a the general solution to

\[ t^2 y''(t) - t(t + 2)y' + (t + 2)y = 2t^3, \quad t > 0 \]

given that \( y_1(t) = t \) and \( y_2(t) = te^t \) are solutions to the homogeneous problem.
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Solution

First, we write the non-homogeneous problem in the standard form from FACT 1. Dividing out by \( t^2 \) yeilds

\[ y''(t) - \frac{t(t + 2)}{t^2}y' + \frac{(t + 2)}{t^2}y = 2t \quad t > 0 \]
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Here \( p(t) = \frac{t(t + 2)}{t^2} \), \( q(t) = \frac{(t + 2)}{t^2} \) and \( g(t) = 2t \) are continuous for \( t > 0 \) so we can apply FACT 1.
Example

We need to check that $y_1$ and $y_2$ form a fundamental set of solutions by showing that the Wronskian is non-zero. Indeed, $y_1(t) = t$, $y_2(t) = te^t$ and $g(t) = 2t$, therefore

$$W[y_1, y_2] = y_2'y_1 - y_1'y_2 = (te^t + e^t)t - (te^t) = t^2e^t.$$
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Using the variation of parameters technique, $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where
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Using the variation of parameters technique, \( y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \) where

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u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)} \, dt = -\int \frac{(te^t)(2t)}{t^2e^t} \, dt = -\int 2 \, dt = -2t
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$$u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)} = \int \frac{t(2t)}{t^2 e^t} \, dt = \int 2e^{-t} \, dt = -2e^{-t}$$

Therefore the general solution is $y(t) = c_1t + c_2(te^t) + (-2t) + (-2e^{-t})$. Variation of Parameters

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W[y_1, y_2] = y_2' y_1 - y_1' y_2 = (te^t + e^t)t - (te^t) = t^2 e^t.
\]

Using the variation of parameters technique, \(y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)\) where

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u_1(t) = - \int \frac{y_2(t)g(t)}{W[y_1, y_2](t)} \, dt = - \int \frac{(te^t)(2t)}{t^2 e^t} \, dt = - \int 2 \, dt = -2t
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Therefore the general solution is

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y(t) = c_1 t + c_2 (te^t) + (-2t) t + (-2e^{-t})(te^t)
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