Variation of Parameters

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Solve a second order ODE of the form

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FACT 1

If $p(t), q(t)$ and $g(t)$ are continuous on an open interval and if y_1 and y_2 form a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$
then the particular solution of the general non-homogeneous problem is

$$y_p(t) = \left(-\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\, dt \right) y_1(t) + \left(\int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}\, dt \right) y_2(t)$$

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and the general solution is

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)}$$

Example

Find a the general solution to

$$t^2y''(t) - t(t+2)y' + (t+2)y = 2t^3, \quad t > 0$$

given that $y_1(t) = t$ and $y_2(t) = te^t$ are solutions to the homogeneous problem.

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Solution

First, we write the non-homogeneous problem in the standard from from FACT 1. Dividing out by t^2 yeilds

$$y''(t) - \frac{t(t+2)}{t^2} y' + \frac{(t+2)}{t^2} y = 2t \quad t > 0$$

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Here $p(t) = \dfrac{t(t+2)}{t^2}, q(t) = \dfrac{(t+2)}{t^2}$ and $g(t) = 2t$ are continuous for $t > 0$ so we can apply FACT 1.

Example

We need to check that y_1 and y_2 form a fundamental set of solutions by showing that the Wronskian is non-zero. Indeed, $y_1(t) = t$, $y_2(t) = te^t$ and $g(t) = 2t$, therefore

$$W[y_1, y_2] = y_2'y_1 - y_1'y_2 = (te^t + e^t)t - (te^t) = t^2e^t.$$

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$$u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\, dt = -\int \frac{(te^t)(2t)}{t^2 e^t}\, dt = -\int 2\, dt = -2t$$

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Using the variation of parameters technique, $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

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Therefore the general solution is

$$\boxed{y(t) = c_1 t + c_2(te^t) + (-2t)t + (-2e^{-t})(te^t)}$$