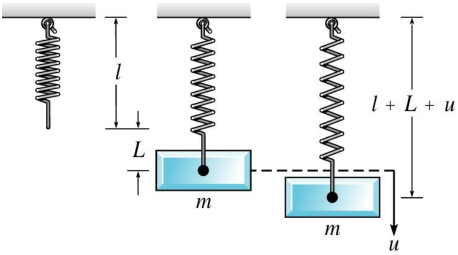
Variation of Parameters

Mechanical Vibrations

Mass –Spring motion



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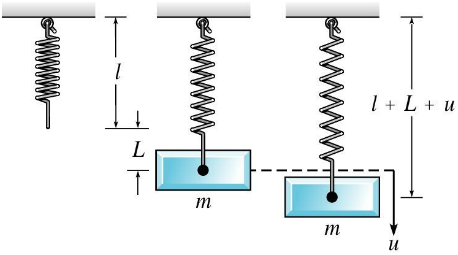

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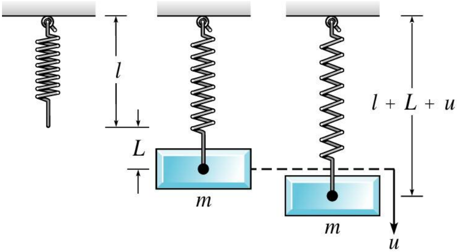

We consider the motion of a mass m on a vertical spring of length l with a small elongation L and let $u(t)$ be the displacement of the mass. (measured positive in a downward direction).

- Second order ODEs with constant coefficients can model a vibrating of the spring-mass system

Spring-mass system

Forces - static case

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- Gravitational force

$$F_g = mg$$

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- Spring force (Hookes Law with spring constant k)

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- In the static case, the spring is in equilibrium so

$$F_g + F_s = 0 \Longrightarrow mg - kL = 0$$

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- **Spring force** (Hooke's Law with spring constant k)

$$\boxed{F_s = -k(L + u)}$$

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  - Let $F_e = f(t)$

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and thus

$$\boxed{mu''(t) + \gamma u'(t) + ku(t) = f(t)}$$

where m, γ and k are constants.

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$$u(0) = u_0, u'(0) = v_0$$

- The initial conditions correspond to initial position and velocity