Forced Periodic Vibrations

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where
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  - $u_c(t)$ is a solution to the homogeneous problem (damped free system)
  - Therefore $\lim_{t\to\infty} u_c(t) = 0$

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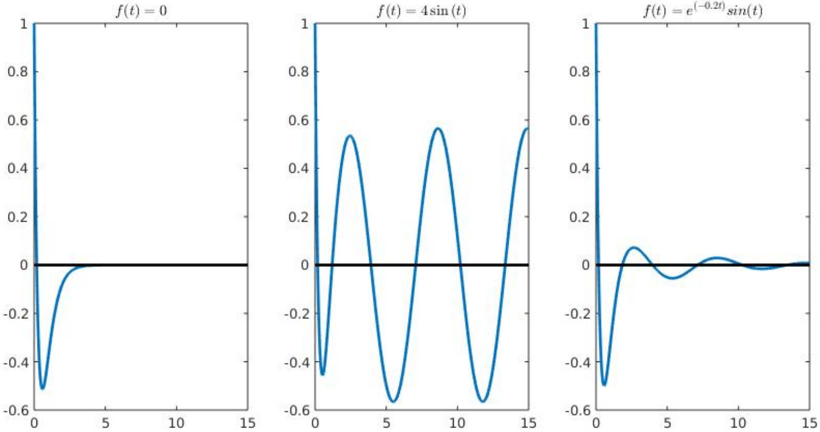
Example 1

A mass spring system with $k = 6N/M, \gamma = 5kg/s$ and $m = 1kg$. The mass is pulled down $1m$ and given an upward velocity of $8m/s$. Find the solution for the following

1. $f(t) = 0$
2. $f(t) = 4\sin(t)$
3. $f(t) = 4e^{-0.2t}\sin(t)$

Example 1 - solutions



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- i.e solve

$$\boxed{mu'' + ku = F_0 cos(\omega_0 t)}$$

Example: $u'' + 4u = \cos(2t), \quad u(0) = 0, u'(0) = 0$

- Solve the homogeneous problem

The characteristic polynomial is $r^2 + 4 = 0 \Longrightarrow r = \pm 2\boldsymbol{i}$. Therefore

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- *Note: we add a factor of t because $2\boldsymbol{i}$ is a solution to the characteristic polynomial*

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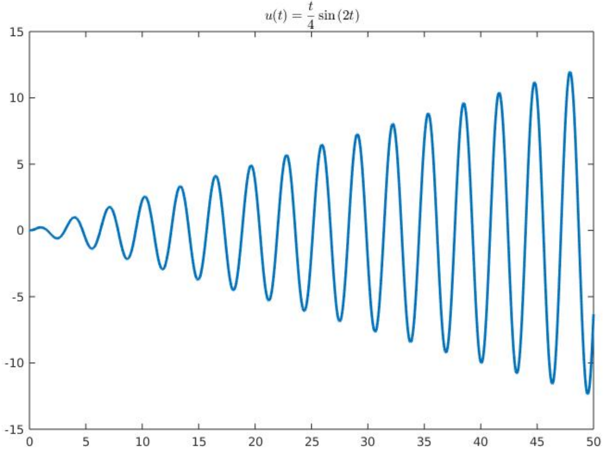

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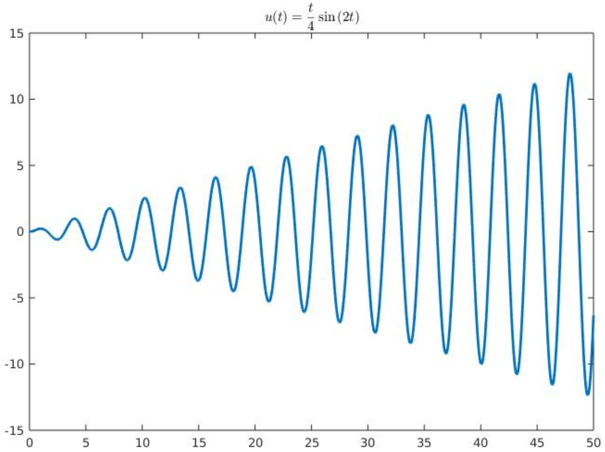

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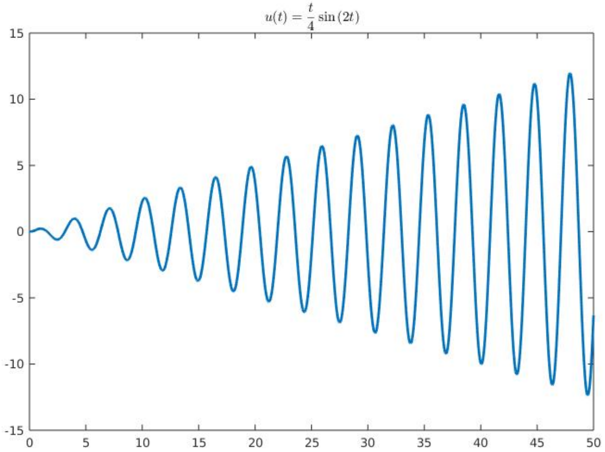

- The solution oscillates with increasing amplitude
- This is an example of resonance

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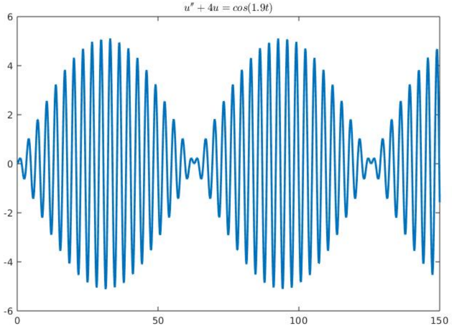

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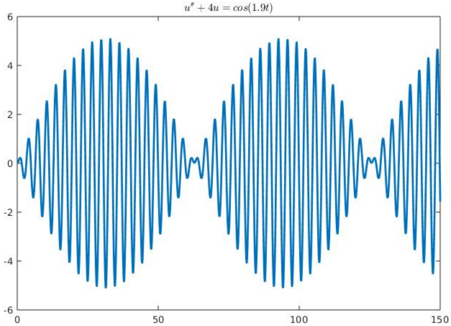

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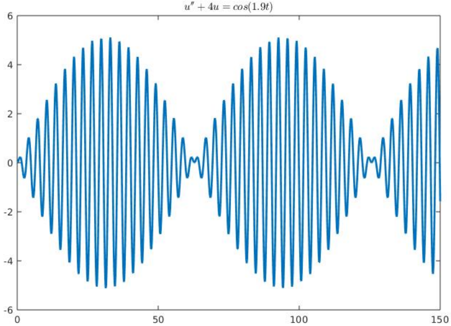

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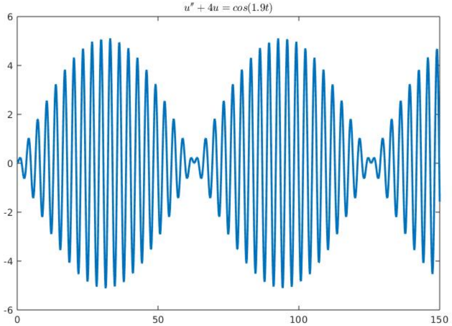

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- e.g. tuning forks with almost the same frequency