Laplace Transforms

Motivation

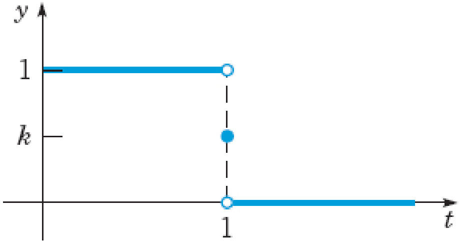
- Solve second order ODEs with discontinuous forcing functions

Motivation

- Solve second order ODEs with discontinuous forcing functions
- WHY?

Motivation

- Solve second order ODEs with discontinuous forcing functions
- WHY? These are common in scientific and engineering applications - think about a forcing pulse that turns on and off, e.g.



Improper integrals

An improper integral is defined as

$$\int_a^\infty f(t)\,dt = \lim_{b->\infty} \int_a^b f(t)\,dt, b > 0$$

Improper integrals

An improper integral is defined as

$$\int_a^\infty f(t)\,dt = \lim_{b->\infty} \int_a^b f(t)\,dt, b > 0$$

If $\lim_{b->\infty} \int_a^b f(t)\,dt$ exists, the improper integral is said to **converge**

Improper integrals - example

$$\int_1^\infty \frac{1}{t^p}\,dt = \lim_{b\to\infty}\int_1^b \frac{1}{t^p}\,dt$$

Improper integrals - example

$$\int_1^\infty \frac{1}{t^p}\,dt = \lim_{b\to\infty}\int_1^b \frac{1}{t^p}\,dt$$

$$= \lim_{b\to\infty}\left[\frac{t^{-p+1}}{-p+1}\bigg|_1^b\right]$$

Improper integrals - example

$$\int_1^\infty \frac{1}{t^p}\,dt = \lim_{b\to\infty}\int_1^b \frac{1}{t^p}\,dt$$
$$= \lim_{b\to\infty}\left[\frac{t^{-p+1}}{-p+1}\bigg|_1^b\right]$$
$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(b^{1-p}-1\right)\right]$$

Improper integrals - example

$$\int_1^\infty \frac{1}{t^p}\,dt = \lim_{b\to\infty}\int_1^b \frac{1}{t^p}\,dt$$
$$= \lim_{b\to\infty}\left[\frac{t^{-p+1}}{-p+1}\bigg|_1^b\right]$$
$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(b^{1-p}-1\right)\right]$$

- As $b\to\infty, b^{1-p}\to 0$ if $p>1$ therefore $\int_1^\infty \frac{1}{t^p}\,dt$ converges.

Improper integrals - example

$$\int_1^\infty \frac{1}{t^p}\,dt = \lim_{b\to\infty}\int_1^b \frac{1}{t^p}\,dt$$
$$= \lim_{b\to\infty}\left[\frac{t^{-p+1}}{-p+1}\bigg|_1^b\right]$$
$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(b^{1-p}-1\right)\right]$$

- As $b \to \infty, b^{1-p} \to 0$ if $p > 1$ therefore $\displaystyle\int_1^\infty \frac{1}{t^p}\,dt$ converges.
- If $p < 1, b^{1-p} \to \infty$ therefore $\displaystyle\int_1^\infty \frac{1}{t^p}\,dt$ diverges

Piecewise functions

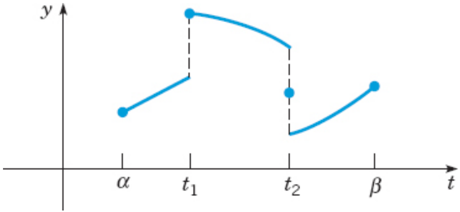
- $f(t)$ is piecewise continuous on $[0, \infty)$ if it is continuous except possibly at a finite number of points t_k.

Piecewise functions

- $f(t)$ is piecewise continuous on $[0, \infty)$ if it is continuous except possibly at a finite number of points t_k.
- At each discontinuity, the left and right-hand side limits exist

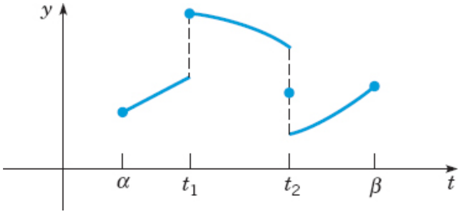
Piecewise functions

- $f(t)$ is piecewise continuous on $[0, \infty)$ if it is continuous except possibly at a finite number of points t_k.
- At each discontinuity, the left and right-hand side limits exist
- For example:



Piecewise functions

- $f(t)$ is piecewise continuous on $[0, \infty)$ if it is continuous except possibly at a finite number of points t_k.
- At each discontinuity, the left and right-hand side limits exist
- For example:



- In this case

$$\int_{\alpha}^{\beta} f(t)\,dt = \int_{\alpha}^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^{\alpha} f(t)\,dt$$

Convergence/Divergence

FACT

If f is piecewise continuous on (a, ∞) and if $|f(t)| \leq g(t)$ for $t > M > 0$ and if $\int_M^\infty g(t)\, dt$ converges, then $\int_a^\infty f(t)$ converges.

Convergence/Divergence

FACT

If f is piecewise continuous on (a,∞) and if $|f(t)| \leq g(t)$ for $t > M > 0$ and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)$ converges. On the other hand if $f(t) \geq g(t) > 0$ for $t \geq M$ and if $\int_M^\infty g(t)\,dt$ diverges then $\int_a^\infty f(t)\,dt$ diverges.

Exponential order

Exponential order

$f(t)$ is said to be of exponential order if there exits M, a and T such that

$$|f(t)| \leq Me^{at} \text{ for } t > T$$

Exponential order

Exponential order

$f(t)$ is said to be of exponential order if there exits M, a and T such that

$$|f(t)| \leq Me^{at} \text{ for } t > T$$

- Most engineering and scientific functions behave in this manner

Laplace Transform

- Let $f(t)$ be piecewise continuous and of exponential order, the Laplace transform is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Laplace Transform

- Let $f(t)$ be piecewise continuous and of exponential order, the Laplace transform is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \equiv F(s)$$

Laplace Transform - constants

- Laplace transform of 1:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt$$

Laplace Transform - constants

- Laplace transform of 1:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{B\to\infty}\int_0^B e^{-st}\,dt$$

Laplace Transform - constants

- Laplace transform of 1:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{B\to\infty}\int_0^B e^{-st}\,dt = \lim_{B\to\infty}\left[\frac{e^{-st}}{-s}\Big|_0^B\right]$$

Laplace Transform - constants

- Laplace transform of 1:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{B\to\infty}\int_0^B e^{-st}\,dt = \lim_{B\to\infty}\left[\frac{e^{-st}}{-s}\Bigg|_0^B\right]$$

$$= \lim_{B\to\infty}\left[\frac{e^{-sB}}{-s} - \frac{1}{-s}\right] = \frac{1}{s}, s > 0$$

Laplace Transform - Exponential functions

- Laplace transform of e^{at}:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt$$

Laplace Transform - Exponential functions

- Laplace transform of e^{at}:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt$$

Laplace Transform - Exponential functions

- Laplace transform of e^{at}:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

Laplace Transform - Exponential functions

- Laplace transform of e^{at}:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \lim_{B\to\infty}\int_0^B e^{-st}(e^{at})\,dt$$

Laplace Transform - Exponential functions

- Laplace transform of e^{at}:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \lim_{B\to\infty}\int_0^B e^{-st}(e^{at})\,dt$$
$$= \lim_{B\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\bigg|_0^B\right]$$

Laplace Transform - Exponential functions

- Laplace transform of e^{at}:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \lim_{B\to\infty}\int_0^B e^{-st}(e^{at})\,dt$$

$$= \lim_{B\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\bigg|_0^B\right] = \lim_{B\to\infty}\left[\frac{e^{-(s-a)B}}{-(s-a)} - \left(\frac{1}{-(s-a)}\right)\right]$$

Laplace Transform - Exponential functions

- Laplace transform of e^{at}:

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \lim_{B\to\infty}\int_0^B e^{-st}(e^{at})\,dt \\ &= \lim_{B\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\bigg|_0^B\right] = \lim_{B\to\infty}\left[\frac{e^{-(s-a)B}}{-(s-a)} - \left(\frac{1}{-(s-a)}\right)\right] \\ &= \frac{1}{s-a}, s > a\end{aligned}$$

Laplace Transform - Trigonometric function

- Laplace transform of $\sin(bt)$ or $\cos(bt)$

Laplace Transform - Trigonometric function

- Laplace transform of $\sin(bt)$ or $\cos(bt)$
- Recall Euler's formula:

$$e^{\boldsymbol{i}\theta} = \cos(\theta) + \boldsymbol{i}\sin(\theta)$$

Laplace Transform - Trigonometric function

- Laplace transform of $\sin(bt)$ or $\cos(bt)$
- Recall Euler's formula:

$$e^{\boldsymbol{i}\theta} = \cos(\theta) + \boldsymbol{i}\sin(\theta) \Longrightarrow e^{\boldsymbol{i}bt} = \cos(bt) + \boldsymbol{i}\sin(bt)$$

Laplace Transform - Trigonometric function

- Laplace transform of $\sin(bt)$ or $\cos(bt)$
- Recall Euler's formula:

$$e^{\boldsymbol{i}\theta} = \cos(\theta) + \boldsymbol{i}\sin(\theta) \Longrightarrow e^{\boldsymbol{i}bt} = \cos(bt) + \boldsymbol{i}\sin(bt)$$

- We compute Laplace transform of $e^{\boldsymbol{i}bt}$

Laplace Transform - Trigonometric function

- Laplace transform of $e^{\boldsymbol{i}bt}$

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \int_0^{\infty} e^{-st}(e^{\boldsymbol{i}bt})\, dt$$

Laplace Transform - Trigonometric function

- Laplace transform of $e^{\boldsymbol{i}bt}$

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \int_0^\infty e^{-st}(e^{\boldsymbol{i}bt})\,dt = \int_0^\infty e^{-(s-\boldsymbol{i}b)t}\,dt$$

Laplace Transform - Trigonometric function

- Laplace transform of $e^{\boldsymbol{i}bt}$

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \int_0^\infty e^{-st}(e^{\boldsymbol{i}bt})\,dt = \int_0^\infty e^{-(s-\boldsymbol{i}b)t}\,dt$$

$$= \lim_{B \longrightarrow \infty} \int_0^B e^{-(s-\boldsymbol{i}b)t}\,dt$$

Laplace Transform - Trigonometric function

- Laplace transform of $e^{\boldsymbol{i}bt}$

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \int_0^\infty e^{-st}(e^{\boldsymbol{i}bt})\,dt = \int_0^\infty e^{-(s-\boldsymbol{i}b)t}\,dt$$

$$= \lim_{B\longrightarrow\infty}\int_0^B e^{-(s-\boldsymbol{i}b)t}\,dt = \lim_{B\to\infty}\left[\frac{e^{-(s-\boldsymbol{i}b)t}}{-(s-\boldsymbol{i}b)}\Big|_0^B\right]$$

Laplace Transform - Trigonometric function

- Laplace transform of $e^{\boldsymbol{i}bt}$

$$
\begin{aligned}
\mathcal{L}\{e^{\boldsymbol{i}bt}\} &= \int_0^\infty e^{-st}(e^{\boldsymbol{i}bt})\,dt = \int_0^\infty e^{-(s-\boldsymbol{i}b)t}\,dt \\
&= \lim_{B\longrightarrow\infty}\int_0^B e^{-(s-\boldsymbol{i}b)t}\,dt = \lim_{B\to\infty}\left[\frac{e^{-(s-\boldsymbol{i}b)t}}{-(s-\boldsymbol{i}b)}\Bigg|_0^B\right] \\
&= \lim_{B\to\infty}\left[\frac{e^{-(s-\boldsymbol{i}b)}B}{-(s-\boldsymbol{i}b)} + \frac{1}{s-\boldsymbol{i}b}\right]
\end{aligned}
$$

Laplace Transform - Trigonometric function

- Laplace transform of $e^{\boldsymbol{i}bt}$

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \int_0^\infty e^{-st}(e^{\boldsymbol{i}bt})\,dt = \int_0^\infty e^{-(s-\boldsymbol{i}b)t}\,dt$$

$$= \lim_{B \longrightarrow \infty} \int_0^B e^{-(s-\boldsymbol{i}b)t}\,dt = \lim_{B\to\infty}\left[\frac{e^{-(s-\boldsymbol{i}b)t}}{-(s-\boldsymbol{i}b)}\Bigg|_0^B\right]$$

$$= \lim_{B\to\infty}\left[\frac{e^{-(s-\boldsymbol{i}b)}B}{-(s-\boldsymbol{i}b)} + \frac{1}{s-\boldsymbol{i}b}\right] = \frac{1}{s-\boldsymbol{i}b}$$

Laplace Transform - Trigonometric function

- Laplace transform of $e^{\boldsymbol{i}bt}$

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \int_0^\infty e^{-st}(e^{\boldsymbol{i}bt})\,dt = \int_0^\infty e^{-(s-\boldsymbol{i}b)t}\,dt$$
$$= \lim_{B\longrightarrow\infty}\int_0^B e^{-(s-\boldsymbol{i}b)t}\,dt = \lim_{B\to\infty}\left[\frac{e^{-(s-\boldsymbol{i}b)t}}{-(s-\boldsymbol{i}b)}\bigg|_0^B\right]$$
$$= \lim_{B\to\infty}\left[\frac{e^{-(s-\boldsymbol{i}b)}B}{-(s-\boldsymbol{i}b)} + \frac{1}{s-\boldsymbol{i}b}\right] = \frac{1}{s-\boldsymbol{i}b}$$

- Observe that

$$\frac{1}{s-\boldsymbol{i}b}\cdot\frac{(s+\boldsymbol{i}b)}{(s+\boldsymbol{i}b)} = \frac{s+\boldsymbol{i}b}{s^2+b^2}$$

Laplace Transform - Trigonometric function

- Laplace transform of $e^{\boldsymbol{i}bt}$

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \int_0^\infty e^{-st}(e^{\boldsymbol{i}bt})\,dt = \int_0^\infty e^{-(s-\boldsymbol{i}b)t}\,dt$$

$$= \lim_{B\longrightarrow\infty}\int_0^B e^{-(s-\boldsymbol{i}b)t}\,dt = \lim_{B\to\infty}\left[\left.\frac{e^{-(s-\boldsymbol{i}b)t}}{-(s-\boldsymbol{i}b)}\right|_0^B\right]$$

$$= \lim_{B\to\infty}\left[\frac{e^{-(s-\boldsymbol{i}b)}B}{-(s-\boldsymbol{i}b)} + \frac{1}{s-\boldsymbol{i}b}\right] = \frac{1}{s-\boldsymbol{i}b}$$

- Observe that

$$\frac{1}{s-\boldsymbol{i}b}\cdot\frac{(s+\boldsymbol{i}b)}{(s+\boldsymbol{i}b)} = \frac{s+\boldsymbol{i}b}{s^2+b^2} = \frac{s}{s^2+b^2} + \boldsymbol{i}\frac{b}{s^2+b^2}$$

Laplace Transform - Trigonometric function

- Thus we have

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \mathcal{L}\{\cos(bt) + \boldsymbol{i}\sin(bt)\} = \frac{s}{s^2+b^2} + \boldsymbol{i}\frac{b}{s^2+b^2}$$

Laplace Transform - Trigonometric function

- Thus we have

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \mathcal{L}\{\cos(bt) + \boldsymbol{i}\sin(bt)\} = \frac{s}{s^2+b^2} + \boldsymbol{i}\frac{b}{s^2+b^2}$$

- Therefore

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, s > 0$$

Laplace Transform - Trigonometric function

- Thus we have

$$\mathcal{L}\{e^{\boldsymbol{i}bt}\} = \mathcal{L}\{\cos(bt) + \boldsymbol{i}\sin(bt)\} = \frac{s}{s^2 + b^2} + \boldsymbol{i}\frac{b}{s^2 + b^2}$$

- Therefore

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2}, s > 0$$

$$\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2}, s > 0$$

Laplace Transform - Piecewise functions

- Let

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ k, & t = 1 \\ 0, & t > 1 \end{cases}$$

Laplace Transform - Piecewise functions

- Let

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ k, & t = 1 \\ 0, & t > 1 \end{cases}$$

- Laplace transform of $f(t)$

Laplace Transform - Piecewise functions

- Let

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ k, & t = 1 \\ 0, & t > 1 \end{cases}$$

- Laplace transform of $f(t)$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Laplace Transform - Piecewise functions

- Let

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ k, & t = 1 \\ 0, & t > 1 \end{cases}$$

- Laplace transform of $f(t)$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, dt = \frac{e^{-st}}{-s}\Big|_0^1 = \frac{1 - e^{-s}}{s}$$

Laplace Transform - Linearity

- The Laplace transform is linear, i.e:

Laplace Transform - Linearity

- The Laplace transform is linear, i.e:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Laplace Transform - Linearity

- The Laplace transform is linear, i.e:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

- This makes sense because the transform is defined by an integral