Laplace Transforms - Solutions of IVPs

Laplace transform of $y = f'(t)$

If f is piecewise continuous and of exponential order, the Laplace transform of the derivative is

Laplace transform of $y = f'(t)$

If f is piecewise continuous and of exponential order, the Laplace transform of the derivative is

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\, dt$$

Laplace transform of $y = f'(t)$

If f is piecewise continuous and of exponential order, the Laplace transform of the derivative is

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t) dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\, dt \\ &= (0 - f(0)) + s \int_0^\infty e^{-st} f(t)\, dt\end{aligned}$$

Laplace transform of $y = f'(t)$

If f is piecewise continuous and of exponential order, the Laplace transform of the derivative is

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t) dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\, dt \\ &= (0 - f(0)) + s\int_0^\infty e^{-st} f(t)\, dt \\ &= s\mathcal{L}\{f(t)\} - f(0)\end{aligned}$$

Laplace transform of $y = f'(t)$

If f is piecewise continuous and of exponential order, the Laplace transform of the derivative is

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^{\infty} e^{-st} f'(t) dt = e^{-st} f(t)\Big|_0^{\infty} - \int_0^{\infty} (-se^{-st}) f(t)\, dt \\ &= (0 - f(0)) + s\int_0^{\infty} e^{-st} f(t)\, dt \\ &= s\mathcal{L}\{f(t)\} - f(0)\end{aligned}$$

where $s > a$

Laplace transform of $y = f^{(n)}(t)$

Suppose $f, f', \cdots, f^{(n-1)}$ are continuous and of exponential order and $f^{(n)}$ is piecewise continuous. Then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Laplace transform - solving a simple ODE

$$y' + 2y = e^{-3t}, y(0) = 1$$

Solution

Laplace transform - solving a simple ODE

$$y' + 2y = e^{-3t}, y(0) = 1$$

Solution

Step. 1 Take the Laplace transform on both sides

Laplace transform - solving a simple ODE

$$y' + 2y = e^{-3t}, y(0) = 1$$

Solution

Step. 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{e^{-3t}\}$$

Laplace transform - solving a simple ODE

$$y' + 2y = e^{-3t}, y(0) = 1$$

Solution

Step. 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{e^{-3t}\} = \frac{1}{s-(-3)} = \frac{1}{s+3}$$

Laplace transform - solving a simple ODE

$$y' + 2y = e^{-3t}, y(0) = 1$$

Solution

Step. 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{e^{-3t}\} = \frac{1}{s-(-3)} = \frac{1}{s+3}$$

The left hand needs a bit more work:

Laplace transform - solving a simple ODE

$$y' + 2y = e^{-3t}, y(0) = 1$$

Solution

Step. 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{e^{-3t}\} = \frac{1}{s-(-3)} = \frac{1}{s+3}$$

The left hand needs a bit more work:

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = s\mathcal{L}\{y\} - y(0) + 2\mathcal{L}\{y\}$$

Laplace transform - solving a simple ODE

$$y' + 2y = e^{-3t}, y(0) = 1$$

Solution

Step. 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{e^{-3t}\} = \frac{1}{s-(-3)} = \frac{1}{s+3}$$

The left hand needs a bit more work:

$$\begin{aligned}\mathcal{L}\{y' + 2y\} = \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= s\mathcal{L}\{y\} - y(0) + 2\mathcal{L}\{y\} \\ &= (s+2)\mathcal{L}\{y\} - 1\end{aligned}$$

Laplace transform - solving a simple ODE

Solution

Combining we obtain $(s+2)\mathcal{L}\{y\} - 1 = \dfrac{1}{s+3}$ therefore

Laplace transform - solving a simple ODE

Solution

Combining we obtain $(s+2)\mathcal{L}\{y\} - 1 = \frac{1}{s+3}$ therefore

$$(s+2)\mathcal{L}\{y\} = \frac{1}{s+3} + 1 = \frac{s+4}{s+3}$$

Laplace transform - solving a simple ODE

Solution

Combining we obtain $(s+2)\mathcal{L}\{y\} - 1 = \dfrac{1}{s+3}$ therefore

$$(s+2)\mathcal{L}\{y\} = \frac{1}{s+3} + 1 = \frac{s+4}{s+3}$$

Step 2 . This means we have $\mathcal{L}\{y\} = \dfrac{s+4}{(s+3)(s+2)}$.

Laplace transform - solving a simple ODE

Solution

Combining we obtain $(s+2)\mathcal{L}\{y\} - 1 = \frac{1}{s+3}$ therefore

$$(s+2)\mathcal{L}\{y\} = \frac{1}{s+3} + 1 = \frac{s+4}{s+3}$$

Step 2 . This means we have $\mathcal{L}\{y\} = \frac{s+4}{(s+3)(s+2)}$.

To obtain the solution to the ODE, find the **inverse Laplace transform:**

$$y = \mathcal{L}^{-1}\left\{\frac{s+4}{(s+3)(s+2)}\right\}$$

Laplace transform - solving a simple ODE

Solution

- Perform a partial fraction decomposition

Laplace transform - solving a simple ODE

Solution

- Perform a partial fraction decomposition

$$\frac{s+4}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$$

Laplace transform - solving a simple ODE

Solution

- Perform a partial fraction decomposition

$$\frac{s+4}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2} = \frac{A(s+2)+B(s+3)}{(s+3)(s+2)}$$

Laplace transform - solving a simple ODE

Solution

- Perform a partial fraction decomposition

$$\frac{s+4}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2} = \frac{A(s+2)+B(s+3)}{(s+3)(s+2)}$$
$$= \frac{(A+B)s + (2A+3B)}{(s+3)(s+2)}$$

Laplace transform - solving a simple ODE

Solution

- Perform a partial fraction decomposition

$$\frac{s+4}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2} = \frac{A(s+2)+B(s+3)}{(s+3)(s+2)}$$
$$= \frac{(A+B)s + (2A+3B)}{(s+3)(s+2)}$$

- Therefore

$$\begin{aligned} A+B &= 1 \\ 2A+3B &= 4 \end{aligned}$$

Laplace transform - solving a simple ODE

Solution

- Perform a partial fraction decomposition

$$\frac{s+4}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2} = \frac{A(s+2)+B(s+3)}{(s+3)(s+2)}$$
$$= \frac{(A+B)s+(2A+3B)}{(s+3)(s+2)}$$

- Therefore

$$\begin{aligned} A+B &= 1 \\ 2A+3B &= 4 \end{aligned}$$

- Solving yeilds $A = -1$ and $B = 2$,

Laplace transform - solving a simple ODE

Solution

- Perform a partial fraction decomposition

$$\frac{s+4}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2} = \frac{A(s+2)+B(s+3)}{(s+3)(s+2)}$$
$$= \frac{(A+B)s+(2A+3B)}{(s+3)(s+2)}$$

- Therefore

$$\begin{aligned} A+B &= 1 \\ 2A+3B &= 4 \end{aligned}$$

- Solving yeilds $A=-1$ and $B=2$, Therefore

$$\frac{s+4}{(s+3)(s+2)} = \frac{-1}{s+3} + \frac{2}{s+2}$$

Laplace transform - solving a simple ODE

Solution

- Finally, we can now compute the inverse Laplace transform to find the solution y

Laplace transform - solving a simple ODE

Solution

- Finally, we can now compute the inverse Laplace transform to find the solution y

$$y = \mathcal{L}^{-1}\left\{\frac{s+4}{(s+3)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+2}\right\}$$

Laplace transform - solving a simple ODE

Solution

- Finally, we can now compute the inverse Laplace transform to find the solution y

$$
\begin{aligned}
y = \mathcal{L}^{-1}\left\{\frac{s+4}{(s+3)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{-1}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+2}\right\} \\
&= -1\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}
\end{aligned}
$$

Laplace transform - solving a simple ODE

Solution

- Finally, we can now compute the inverse Laplace transform to find the solution y

$$
\begin{aligned}
y = \mathcal{L}^{-1}\left\{\frac{s+4}{(s+3)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{-1}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+2}\right\} \\
&= -1\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\
&= -e^{-3t} + 2e^{-2t}
\end{aligned}
$$

Laplace transform - second order ODEs

Solve

$$y'' + 4y' - 5y = te^t \quad y(0) = 1, y'(0) = 0$$

Laplace transform - second order ODEs

Solve

$$y'' + 4y' - 5y = te^t \quad y(0) = 1, y'(0) = 0$$

STEP 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{te^t\}$$

Laplace transform - second order ODEs

Solve

$$y'' + 4y' - 5y = te^t \quad y(0) = 1, y'(0) = 0$$

STEP 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$$

Laplace transform - second order ODEs

Solve

$$y'' + 4y' - 5y = te^t \quad y(0) = 1, y'(0) = 0$$

STEP 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$$

STEP 2 Express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of $\mathcal{L}\{y\}$

Laplace transform - second order ODEs

Solve

$$y'' + 4y' - 5y = te^t \quad y(0) = 1, y'(0) = 0$$

STEP 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$$

STEP 2 Express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of $\mathcal{L}\{y\}$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2\mathcal{L}\{y\} - s$$

Laplace transform - second order ODEs

Solve

$$y'' + 4y' - 5y = te^t \quad y(0) = 1, y'(0) = 0$$

STEP 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$$

STEP 2 Express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of $\mathcal{L}\{y\}$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2\mathcal{L}\{y\} - s$$
$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

Laplace transform - second order ODEs

Solve

$$y'' + 4y' - 5y = te^t \quad y(0) = 1, y'(0) = 0$$

STEP 1 Take the Laplace transform on both sides

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$$

STEP 2 Express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of $\mathcal{L}\{y\}$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2\mathcal{L}\{y\} - s$$
$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = s\mathcal{L}\{y\} - 1$$

Laplace transform - second order ODEs

STEP 3 Substitute back into the ODE and solve for $\mathcal{L}\{y\}$

Laplace transform - second order ODEs

STEP 3 Substitute back into the ODE and solve for $\mathcal{L}\{y\}$

$$\Big[\underbrace{s^2\mathcal{L}\{y\} - s}_{=y''}\Big] + 4\Big[\underbrace{s\mathcal{L}\{y\} - 1}_{=y'}\Big] - 5\mathcal{L}\{y\} = \frac{1}{(s-1)^2}$$

Laplace transform - second order ODEs

STEP 3 Substitute back into the ODE and solve for $\mathcal{L}\{y\}$

$$\big[\underbrace{s^2\mathcal{L}\{y\} - s}_{=y''}\big] + 4\big[\underbrace{s\mathcal{L}\{y\} - 1}_{=y'}\big] - 5\mathcal{L}\{y\} = \frac{1}{(s-1)^2}$$

$$(s^2 + 4s - 5)\mathcal{L}\{y\} = s + 4 + \frac{1}{(s-1)^2}$$

Laplace transform - second order ODEs

STEP 3 Substitute back into the ODE and solve for $\mathcal{L}\{y\}$

$$\Big[\underbrace{s^2\mathcal{L}\{y\} - s}_{=y''}\Big] + 4\Big[\underbrace{s\mathcal{L}\{y\} - 1}_{=y'}\Big] - 5\mathcal{L}\{y\} = \frac{1}{(s-1)^2}$$

$$(s^2 + 4s - 5)\mathcal{L}\{y\} = s + 4 + \frac{1}{(s-1)^2}$$

$$(s+5)(s-1)\mathcal{L}\{y\} = \frac{(s+4)(s-1)^2 + 1}{(s-1)^2}$$

Laplace transform - second order ODEs

STEP 3 Substitute back into the ODE and solve for $\mathcal{L}\{y\}$

$$\Big[\underbrace{s^2\mathcal{L}\{y\} - s}_{=y''}\Big] + 4\Big[\underbrace{s\mathcal{L}\{y\} - 1}_{=y'}\Big] - 5\mathcal{L}\{y\} = \frac{1}{(s-1)^2}$$

$$(s^2 + 4s - 5)\mathcal{L}\{y\} = s + 4 + \frac{1}{(s-1)^2}$$

$$(s+5)(s-1)\mathcal{L}\{y\} = \frac{(s+4)(s-1)^2 + 1}{(s-1)^2} = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2}$$

Laplace transform - second order ODEs

STEP 3 Substitute back into the ODE and solve for $\mathcal{L}\{y\}$

$$\Big[\underbrace{s^2\mathcal{L}\{y\} - s}_{=y''}\Big] + 4\Big[\underbrace{s\mathcal{L}\{y\} - 1}_{=y'}\Big] - 5\mathcal{L}\{y\} = \frac{1}{(s-1)^2}$$

$$(s^2 + 4s - 5)\mathcal{L}\{y\} = s + 4 + \frac{1}{(s-1)^2}$$

$$(s+5)(s-1)\mathcal{L}\{y\} = \frac{(s+4)(s-1)^2 + 1}{(s-1)^2} = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2}$$

The partial fraction decompostion of $\mathcal{L}\{y\}$ has the form

Laplace transform - second order ODEs

STEP 3 Substitute back into the ODE and solve for $\mathcal{L}\{y\}$

$$\Big[\underbrace{s^2\mathcal{L}\{y\} - s}_{=y''}\Big] + 4\Big[\underbrace{s\mathcal{L}\{y\} - 1}_{=y'}\Big] - 5\mathcal{L}\{y\} = \frac{1}{(s-1)^2}$$

$$(s^2 + 4s - 5)\mathcal{L}\{y\} = s + 4 + \frac{1}{(s-1)^2}$$

$$(s+5)(s-1)\mathcal{L}\{y\} = \frac{(s+4)(s-1)^2 + 1}{(s-1)^2} = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2}$$

The partial fraction decompostion of $\mathcal{L}\{y\}$ has the form

$$\frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3} = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}$$

Laplace transform - second order ODEs

Using MATLAB, we can solve to get

$$\mathcal{L}\{y\} = \frac{35}{216}\left(\frac{1}{s+5}\right) + \frac{181}{216}\left(\frac{1}{s-1}\right) - \frac{1}{36}\left(\frac{1}{(s-1)^2}\right) + \frac{1}{6}\left(\frac{1}{(s-1)^3}\right)$$

Laplace transform - second order ODEs

Using MATLAB, we can solve to get

$$\mathcal{L}\{y\} = \frac{35}{216}\left(\frac{1}{s+5}\right) + \frac{181}{216}\left(\frac{1}{s-1}\right) - \frac{1}{36}\left(\frac{1}{(s-1)^2}\right) + \frac{1}{6}\left(\frac{1}{(s-1)^3}\right)$$

STEP 4 Find the inverse Laplace transform of each term to solve for y

Laplace transform - second order ODEs

Using MATLAB, we can solve to get

$$\mathcal{L}\{y\} = \frac{35}{216}\left(\frac{1}{s+5}\right) + \frac{181}{216}\left(\frac{1}{s-1}\right) - \frac{1}{36}\left(\frac{1}{(s-1)^2}\right) + \frac{1}{6}\left(\frac{1}{(s-1)^3}\right)$$

STEP 4 Find the inverse Laplace transform of each term to solve for y

$$y(t) = \frac{35}{216}e^{-5t} + \frac{181}{216}e^{t} - \frac{1}{36}te^{t} + \frac{1}{12}t^2e^{t}$$