Worksheet 03/27 solutions

Find the general solution for

1. $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$ *The solution of the homogeneous problem is* $y_c(t) = c_1 e^t + c_2 t e^t$. *The functions* $y_1(t) = e^t$ *and* $y_2(t) = te^t$ *form a fundamental set with the Wronskain,* $W(y_1, y_2)[t] = e^{2t}$. *Applying the variation of parameters technique, we seek*

$$y_p(t) = u_1(t)e^t + u_2(t)e^{2t}.$$

We compute

$$u_1(t) = -\int \frac{te^t(e^t)}{e^{2t}(1+t^2)}\,dt = -\int \frac{t}{1+t^2}\,dt \underbrace{=}_{U\text{-}sub} -\frac{1}{2}\ln(1+t^2)$$
$$u_2(t) = \int \frac{e^t(e^t)}{e^{2t}(1+t^2)}\,dt = \int \frac{1}{(1+t^2)}\,dt = \arctan(t)$$

The general solution is:

$$y(t) = c_1 e^t + c_2 te^t - \frac{1}{2}e^t \ln(1+t^2) + te^t \arctan(t)$$

2. $4y'' + y = 2\sec(t/2), -\pi < t < \pi$ *The solution of the homogeneous problem is* $y(t) = c_1\cos(t/2) + c_2\sin(t/2)$. *The functions* $y_1(t) = \cos(t/2)$ *and* $y_2(t) = \sin(t/2)$ *form a fundamental set with the Wronskian,* $W(y_1, y_2)[t] = \dfrac{1}{2}$. *Applying the variation of parameters technique, First write the ODE in the standard from*

$$y'' + \frac{y}{4} = \frac{1}{2}\sec(t/2)$$

so that $g(t) = \frac{1}{2}\sec(t/2)$. *Then we seek*

$$y_p(t) = u_1(t)\cos(t/2) + u_2(t)\sin(t/2).$$

We compute

$$u_1(t) = -\int \sin(t/2)[\sec(t/2)] = -\int \frac{\sin(t/2)}{\cos(t/2)}\,dt = 2\ln(cos(t/2)) \text{ here use u-sub with } u = cos(t/2)$$
$$u_2(t) = \int \sin(t/2)[\sec(t/2)]\,dt = \int 1\,dt = t$$

The general solution is:

$$y(t) = c_1\cos(t/2) + c_2\sin(t/2) + 2\cos(t/2)\ln(cos(t/2)) + t\sin(t/2)$$

Additional Reading/ Examples

Section 3.6 pages 142-143 (Example 1)