WORKSHEET 04/24 Solution

Solve $\quad y'' + 6y' + 5y = 3e^{-2t}, \quad y(0) = 1, \quad y'(0) = 1.$

Step 1

Take the Laplace transform on both sides

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 3\mathcal{L}\{e^{-2t}\}$$

Step 2

Recall that

$$\left.\begin{aligned} \mathcal{L}\{y''\} &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2\mathcal{L}\{y\} - s - 1 \\ \mathcal{L}\{y'\} &= s\mathcal{L}\{y\} - y(0) = s\mathcal{L}\{y\} - 1 \end{aligned}\right\}$$

Notice the application of initial conditions.

Plug into ODE

$$\left[s^2\mathcal{L}\{y\} - s - 1\right] + 6\left[s\mathcal{L}\{y\} - 1\right] + 5\mathcal{L}\{y\} = 3 \cdot \frac{1}{s+2}$$

so

from table $\mathcal{L}\{e^{+at}\} = \frac{1}{s-a}$, $a = -2$.

Grouping terms with $\mathcal{L}\{y\}$ yields

$$(s^2 + 6s + 5)\mathcal{L}\{y\}\ (-s - 1 - 6) = \frac{3}{s+2} \Rightarrow (s^2 + 6s + 5)\mathcal{L}\{y\} = \frac{3}{s+2} + (s+7)$$

Notice that $\quad s^2 + 6s + 5 = (s+1)(s+5),\quad$ then

$$(s+1)(s+5)\mathcal{L}\{y\} = \frac{3}{s+2} + (s+7)$$

$$= \frac{3 + (s+2)(s+7)}{(s+2)} = \frac{3 + s^2 + 9s + 14}{(s+2)}$$

Yielding $\quad (s+1)(s+5)\mathcal{L}\{y\} = \dfrac{s^2 + 9s + 17}{(s+2)}\quad$ so that

$$\mathcal{L}\{y\} = \frac{s^2 + 9s + 17}{(s+2)(s+1)(s+5)}.$$

To find y we need to compute $\mathcal{L}^{-1}\left\{\dfrac{s^2 + 9s + 17}{(s+2)(s+1)(s+5)}\right\}$. For that we do a partial fraction decomposition.

$$\frac{s^2+9s+17}{(s+2)(s+1)(s+5)} = \frac{A}{s+2} + \frac{B}{s+5} + \frac{C}{s+1}$$

$$\Rightarrow \frac{s^2+9s+17}{(s+2)(s+1)(s+5)} = \frac{A(s+5)(s+1) + B(s+2)(s+1) + C(s+2)(s+5)}{(s+2)(s+1)(s+5)}$$

Comparing numerators

$$s^2+9s+17 = A(s+5)(s+1) + B(s+2)(s+1) + C(s+2)(s+5)$$

Since the rhs is comprised of linear factors pick

$\underline{s=-1}$ to eliminate A, B so that

$$(-1)^2 + 9(-1) + 17 = C(-1+2)(-1+5)$$

$$1-9+17 = 4C \Rightarrow 4C = 9 \Rightarrow C = \frac{9}{4}$$

$\underline{s=-2}$

$$(-2)^2 + 9(-2) + 17 = A(-2+5)(-2+1)$$

$$4 - 18 + 17 = A(3)(-1) \Rightarrow -3A = 3 \Rightarrow A = -1$$

$\underline{c=-5}$

$$(-5)^2 + 9(-5) + 17 = B(-5+2)(-5+1)$$

$$25 - 45 + 17 = 12B \Rightarrow 12B = -3 \Rightarrow B = -\frac{1}{4}$$

Thus we have

$$\frac{s^2+9s+17}{(s+2)(s+1)(s+5)} = \frac{-1}{s+2} - \frac{1}{4}\left(\frac{1}{s+5}\right) + \frac{9}{4}\left(\frac{1}{s+1}\right)$$

and

$$y = \mathcal{L}^{-1}\left\{\frac{-1}{s+2} - \frac{1}{4}\left(\frac{1}{s+5}\right) + \frac{9}{4}\left(\frac{1}{s+1}\right)\right\}$$

$$= -1\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} + \frac{9}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -e^{-2t} - \frac{1}{4}e^{-5t} + \frac{9}{4}e^{-t}.$$