Initial Value Problems of ODEs
Differential Equations

**Objective:** Solve an equation containing an unknown function and one or more of its derivatives

**Problem:** Find $y(t)$ satisfying:

\[ y'(t) = f(t, y(t)), \quad t \geq t_0 \quad (1) \]
\[ y(t_0) = y_0 \quad (2) \]

- (1)–(2) is called an initial value problem of the ODE $y'(t) = f(t, y)$.
- $y(t)$ is a scalar valued function of $t$. 
Examples

\[ y'(t) = \sin(t) \]
\[ y(\frac{\pi}{3}) = 2 \]

- \( y'(t) = \sin(t) \implies y(t) = -\cos(t) + C \) (general solution)
Examples

\[ y'(t) = \sin(t) \]
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- \( y'(t) = \sin(t) \iff y(t) = -\cos(t) + C \) (general solution)
- Using initial condition \( y\left(\frac{\pi}{3}\right) = 2 \), we have a particular solution by solving for \( C \)
  \[ y(t) = 2.5 - \cos(t) \]
Examples

\[ y'(t) = \lambda y(t) + b(t), \quad t \geq t_0 \text{ and } \lambda \text{ constant} \]
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Solve using the method of integrating factors!

\[ y'(t) - \lambda y(t) = b(t) \]
\[ (y'(t) - \lambda y(t))e^{-\lambda t} = e^{-\lambda t}b(t) \]

Recognize that

\[ (y'(t) - \lambda y(t))e^{\lambda t} = \frac{d}{dt}(y(t)e^{-\lambda t}) \]

then

\[ \frac{d}{dt}(y(t)e^{-\lambda t}) = e^{-\lambda t}b(t) \]
Examples

\[ \frac{d}{dt}(y(t)e^{-\lambda t}) = e^{-\lambda t} b(t) \]

Integrating both sides:

\[ e^{-\lambda t} y(t) = \int_{t_0}^{t} e^{-\lambda s} b(s) \, ds + C \]

so the general solution is

\[ y(t) = e^{\lambda t} \left[ C + \int_{t_0}^{t} e^{-\lambda s} b(s) \, ds \right] = Ce^{\lambda t} + \int_{t_0}^{t} e^{\lambda(t-s)} b(s) \, ds \]
If the displacement is not too large, the force exerted on the mass is proportional to the displacement from the origin.

\[ y''(t) = -ky, \quad k > 0 \]

\[ y(0) = y_0 \]

\[ y'(0) = 0 \]

\[ y(t) = c_1 \sin(\sqrt{k} t) + c_2 \cos(\sqrt{k} t) \]
Euler’s method

\[ y'(t) = f(t, y(t)), \]
\[ y(a) = y_0 \quad a \leq t \leq b \]

- **Step 1:** Divide \([a, b]\) into \(N\) subintervals of size \(h = \frac{b - a}{N}\)

\[ a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b \]
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- **Step 2:** Replace \(y'(t)\) by an approximation (from calc I)

\[ y'(t) = \lim_{{h \to 0}} \frac{y(t + h) - y(t)}{h} \]

so starting with \([t_0, t_1]\),

\[ y'(t_0) \approx \frac{y(t_1) - y(t_0)}{h} = f(t_0, y_0) \]
Euler’s method for $y'(t) = f(t, y(t))$ \quad $a \leq t \leq b$

**t grid:** \hspace{1cm} $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$

- On subinterval $[t_0, t_1]$: replace $y'(t_0)$ by a finite difference:

  \[
  y'(t_0) \approx \frac{y(t_1) - y(t_0)}{h} = f(t_0, y_0) \Rightarrow y_1 = y_0 + hf(t_0, y_0)
  \]
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- On subinterval $[t_1, t_2]$: replace $y'(t_1)$ by a finite difference:
  \[
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Euler’s method for $y'(t) = f(t, y(t)) \quad a \leq t \leq b$

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In general on subinterval $[t_k, t_{k+1}]$: replace $y'(t_k)$ by a finite:

$$y'(t_k) \approx \frac{y(t_{k+1}) - y(t_k)}{h} = f(t_k, y_k) \implies y_{k+1} = y_k + hf(t_k, y_k)$$
Euler’s method - quick derivation

Brook Taylor (1685–1731)

Theorem (Taylor)

Let \( f, f', \ldots f^{(n)} \) be continuous on \([a, b]\) and let \( f^{(n+1)} \) exist for all \( t \) in \((a, b)\). Then there is a number \( \xi \) between \( t \) and \( a \) such that

\[
f(t) = f(a) + (t - a)f'(a) + \frac{(t - a)^2}{2!}f''(a) + \cdots + \frac{(x - a)^n}{n!}f^{(n)}(a) + \frac{(x - a)^{n+1}}{(n + 1)!}f^{(n+1)}(\xi)
\]

\(^1\)http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Taylor.html
Euler’s method - quick derivation

Leonard Euler (1707–1783)

\[ y_{k+1} = y_k + hf(x_k, y_k) \]

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Euler’s method - quick derivation

Taylor’s Theorem

Given the (IVP) \( y'(t) = f(t, y(t)) \), expand \( y(t) \) about \( t_k \) as

\[
y(t) = y(t_k) + (t - t_k)y'(t_k) + \frac{(t - t_k)^2}{2}y''(\xi_k), \quad \xi_k \in [t_k, t_{k+1}]
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$$= y(t_k) + (t - t_k)f(t_k, y(t_k)) + \frac{(t - t_k)^2}{2}y''(\xi_k), \quad \xi_k \in [t_k, t_{k+1}]$$
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\]

Evaluating at \( t = t_{k+1} \) and recalling that \( h = t_{k+1} - t_k \) yields:

\[
y(t_{k+1}) = y(t_k) + hf(t_k, y(t_k)) + \frac{h^2}{2}y''(\xi_k)
\]

Dropping the error term \( \frac{h^2}{2}y''(\xi_k) \)

\[
y_{k+1} = y_k + hf(t_k, y_k), \quad k = 0, 1, 2, \ldots
\]
Euler’s method - works well in some cases

Leonard Euler (1707–1783)

\[ y_{k+1} = y_k + hf(x_k, y_k) \]

Euler’s method is simple and works for some problems BUT is prone to errors and can be unstable for *stiff problems* (MA 428) [Hidden Figures]

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3[http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Euler.html](http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Euler.html)
Euler’s method - modeling the spread of an infection

Modeling the spread of an epidemic (Kermack & McKendrick, Proc Roy, Soc. (1927))

Assumptions

- Population divided into healthy individuals \((H)\), infected individuals \((I)\) and the dead \((D)\)
Euler’s method - modeling the spread of an infection


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- Population divided into **healthy individuals** (H), **infected individuals** (I) and the **dead** (D)
- The epidemic spreads so quickly that changes in population due to birth, death or migration can be ignored.
- The disease is transmitted to healthy individuals at a rate proportional to the product of **healthy** and **infected** people
Euler’s method for systems - modeling the spread of an infection

- Turning our assumptions into equations:
  \[
  \frac{dH}{dt} = -cHI, \quad \frac{dI}{dt} = cHI - mI, \quad \frac{dD}{dt} = mI
  \]
  where \( c \) is the transmission rate and \( m \) is the mortality rate of infected individuals.

- The model can be reduced to a single equation. First divide the \( H \) equation by the \( D \) equation to get
  \[
  \frac{dH}{dD} = -\frac{c}{m}H
  \]
  whose solution is:
Euler’s method for systems - modeling the spread of an infection

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\[
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where \( c \) is the **transmission rate** and \( m \) is the **mortality rate** of infected individuals.

- The model can be reduced to a single equation. First divide the \( H \) equation by the \( D \) equation to get

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\frac{dH}{dD} = -\frac{c}{m}H
\]

whose solution is:

\[
H = H_0 e^{-\frac{c}{m}D}
\]

where \( H_0 \) is the number of healthy individuals.
If $N$ is the size of the population then

$$H + I + D = N \implies I = N - H - D$$

Once $D$ is determined, we can solve for $H$ and $I$ using

$$H = H_0 e^{-\frac{c}{m}D} \quad I = N - H - D$$
Exercise

Write a function
\[ [t\_vals, y\_vals] = \text{approx}\_ode}(f,a,b,\text{initial\_value},n) \]
that takes as input a function defined in a file \( y' = f(t,y) \), defined in \( f.m \) and two values \( a \) and \( b \) indicating the left and right intervals on which the initial value problem is defined and \( n \), the number of Euler steps. Your function should return 2 vectors (\( t\_vals \) contains the values \( t_0, t_1, \cdots, t_N \) and \( y\_vals \) is the solution at those points \( y_0, y_1, \cdots, y_N \)) Test your code on the initial value problem

\[ \frac{dy}{dt} = 1 + \frac{y}{t} \quad 1 \leq t \leq 6, \quad y(1) = 1 \]

whose exact solution is \( y(t) = t(1 + \ln t) \).