# Linear Transformations

# Linear Transformations

A function $\boldsymbol{T}$ from $\mathbb{R}^n \to \mathbb{R}^m$ is called a *linear transformation* if there exists an $m \times n$ matrix $\boldsymbol{A}$ such that

$$\boldsymbol{T}(\vec{x}) = \boldsymbol{A}\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$ satisfying the following:

# Linear Transformations

A function $\boldsymbol{T}$ from $\mathbb{R}^n \to \mathbb{R}^m$ is called a *linear transformation* if there exists an $m \times n$ matrix $\boldsymbol{A}$ such that

$$\boldsymbol{T}(\vec{x}) = \boldsymbol{A}\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$ satisfying the following:

- $\boldsymbol{T}(\vec{v} + \vec{w}) = \boldsymbol{T}(\vec{v}) + \boldsymbol{T}(\vec{w}), \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^n$
- $\boldsymbol{T}(c\vec{v}) = c\boldsymbol{T}(\vec{v}), \quad \forall \vec{v} \in \mathbb{R}^n, c \in \mathbb{R}$

# Linear Transformations

A function $\boldsymbol{T}$ from $\mathbb{R}^n \to \mathbb{R}^m$ is called a *linear transformation* if there exists an $m \times n$ matrix $\boldsymbol{A}$ such that

$$\boldsymbol{T}(\vec{x}) = \boldsymbol{A}\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$ satisfying the following:

- $\boldsymbol{T}(\vec{v} + \vec{w}) = \boldsymbol{T}(\vec{v}) + \boldsymbol{T}(\vec{w}), \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^n$
- $\boldsymbol{T}(c\vec{v}) = c\boldsymbol{T}(\vec{v}), \quad \forall \vec{v} \in \mathbb{R}^n, c \in \mathbb{R}$

*Linear transformations preserve lines, unlike nonlinear transformations that may transform a line segment into a parabolic curve, or ellipse*

# Linear Transformations in 2D

- We focus on $\boldsymbol{T}$ from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

# Linear Transformations in 2D

- We focus on $\boldsymbol{T}$ from $\mathbb{R}^2 \to \mathbb{R}^2$
- $\boldsymbol{A}$ is a $2 \times 2$ matrix and $\vec{v}$ is a $2 \times 1$ column vector.

# Linear Transformations in 2D

- We focus on $\boldsymbol{T}$ from $\mathbb{R}^2 \to \mathbb{R}^2$
- $\boldsymbol{A}$ is a $2 \times 2$ matrix and $\vec{v}$ is a $2 \times 1$ column vector.
- Special examples of linear transformations include:
  1. scaling transformations
  2. rotations
  3. translations

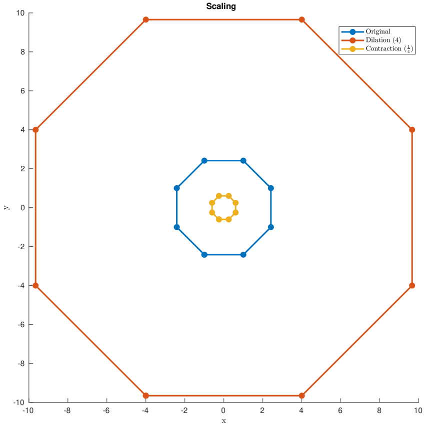
# Scaling Transformations

$$\boldsymbol{T}\colon \mathbb{R}^2 \to \mathbb{R}^2 \text{ defined by } \boldsymbol{T}(\vec{v}) = c\vec{v} \text{ for } c \in (0, \infty)$$

- $c > 1$ - dilation by a factor of $c$
- $c < 1$ - contraction by a factor of $c$
- In matrix form

$$\boldsymbol{T}\left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

# Scaling Transformations



# Rotations

**Rotations** by an angle $\theta$ about the origin where the rotation is measured from the positive $x$-axis in an anticlockwise direction

- In matrix form, the linear transformation can be represented as:

$$\boldsymbol{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

## Reflections

**Reflections** about a line $L$ through the origin, e.g.

- Reflecting a point in $\mathbb{R}^2$ about the $y$-axis:

$$\boldsymbol{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$$

in matrix form

# Reflections

**Reflections** about a line $L$ through the origin, e.g.

- Reflecting a point in $\mathbb{R}^2$ about the $y$-axis:

$$\boldsymbol{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$$

in matrix form

$$\boldsymbol{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

# Reflections

**Reflections** about a line $L$ through the origin, e.g.

- Reflecting a point in $\mathbb{R}^2$ about the $y$-axis:

$$\boldsymbol{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$$

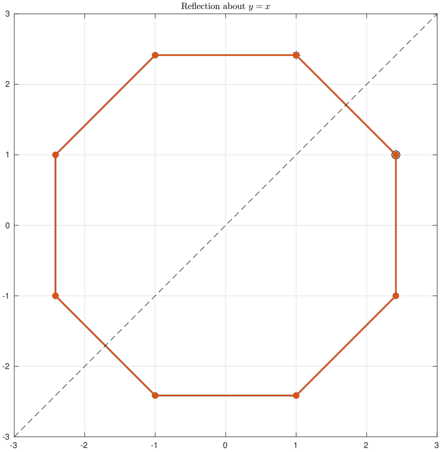
  in matrix form

$$\boldsymbol{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- In general, the transformation corresponding to a reflection about the line $L$ making an angle $\theta$ with the positive $x - axis$ is given by

$$\boldsymbol{A} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad a^2 + b^2 = 1$$

# Reflection



# Shear

- $y$-shear

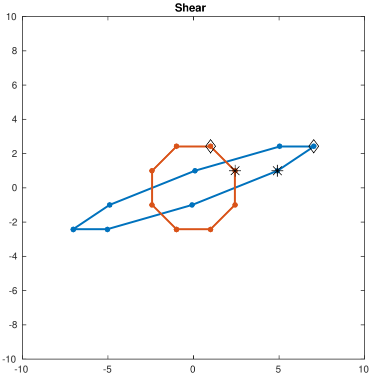
$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

- $x$-shear

$$\boldsymbol{T} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

# $x$-shear

$$\boldsymbol{T} = \begin{bmatrix} 1 & 2.5 \\ 0 & 1 \end{bmatrix}$$



# Compositions of transformations

Given two linear transformations $\boldsymbol{T}$ and $\boldsymbol{S}$ both $\mathbb{R}^2 \to \mathbb{R}^2$ with

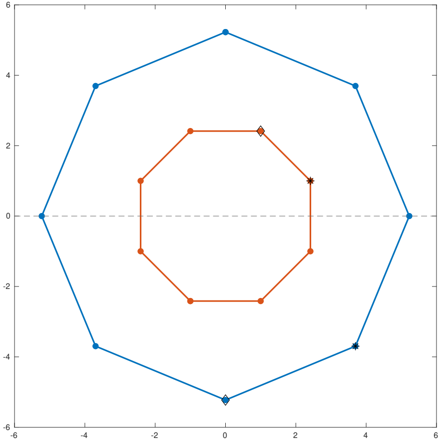
$$\boldsymbol{T}(\vec{v}) = \boldsymbol{A}\vec{v} \text{ and } \boldsymbol{S}(\vec{v}) = \boldsymbol{B}\vec{v} \quad \forall \vec{v} \in \mathbb{R}^2$$

then the composition of the transformation $\boldsymbol{T}$ and $\boldsymbol{S}$, $T \circ S$ $\boldsymbol{AB}$

$$\big(\boldsymbol{T} \circ \boldsymbol{S}\big)(\vec{v}) = \boldsymbol{T}\big(\boldsymbol{S}(\vec{v})\big) = \boldsymbol{T}\big(\boldsymbol{B}\vec{v}\big) = \boldsymbol{AB}\vec{v}$$

# Compositions of transformations

Rotation $\theta = \frac{\pi}{8}$ then reflection about $y = 0$, then dilation by a factor of 2.



## Orthogonal transformations

- A linear transformation $\boldsymbol{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called orthogonal if it preserves the length of vectors:

$$||\boldsymbol{T}(\vec{v})|| = ||\vec{v}||, \quad \forall \vec{v} \in \mathbb{R}^n$$

- If $\boldsymbol{T}(\vec{v}) = \boldsymbol{A}\vec{v}$ is an orthogonal transformation, $\boldsymbol{A}$ is an **orthogonal matrix**

# Orthogonal transformations

- A linear transformation $\boldsymbol{T} : \mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal if it preserves the length of vectors:

$$||\boldsymbol{T}(\vec{v})|| = ||\vec{v}||, \quad \forall \vec{v} \in \mathbb{R}^n$$

- If $\boldsymbol{T}(\vec{v}) = \boldsymbol{A}\vec{v}$ is an orthogonal transformation, $\boldsymbol{A}$ is an **orthogonal matrix**
  1. $||\boldsymbol{A}\vec{v}|| = ||\vec{v}||, \quad \forall \vec{v} \in \mathbb{R}^n$
  2. The columns of $\boldsymbol{A}$ form an orthonormal basis of $\mathbb{R}^n$
  3. $\boldsymbol{A}^T A = \boldsymbol{I}_n$
  4. $A^{-1} = A^T$

# Orthogonal transformations

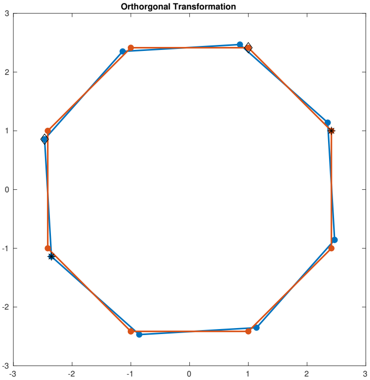
- A linear transformation $\boldsymbol{T} : \mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal if it preserves the length of vectors:

$$||\boldsymbol{T}(\vec{v})|| = ||\vec{v}||, \quad \forall \vec{v} \in \mathbb{R}^n$$

- If $\boldsymbol{T}(\vec{v}) = \boldsymbol{A}\vec{v}$ is an orthogonal transformation, $\boldsymbol{A}$ is an **orthogonal matrix**
  1. $||\boldsymbol{A}\vec{v}|| = ||\vec{v}||, \quad \forall \vec{v} \in \mathbb{R}^n$
  2. The columns of $\boldsymbol{A}$ form an orthonormal basis of $\mathbb{R}^n$
  3. $\boldsymbol{A}^T A = \boldsymbol{I}_n$
  4. $A^{-1} = A^T$
- Orthogonal transformations also preserve dot products of vectors and thus angles are preserved

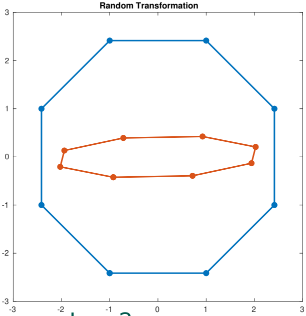
# Random Orthogonal transformations

```
T= orth(rand(2,2))
```



# Random Transformation

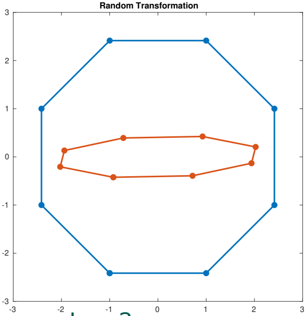
$$M = \begin{bmatrix} 0.8212 & 0.0430 \\ 0.0154 & 0.1690 \end{bmatrix}$$




Can this transformation be undone?

# Random Transformation

$$M = \begin{bmatrix} 0.8212 & 0.0430 \\ 0.0154 & 0.1690 \end{bmatrix}$$

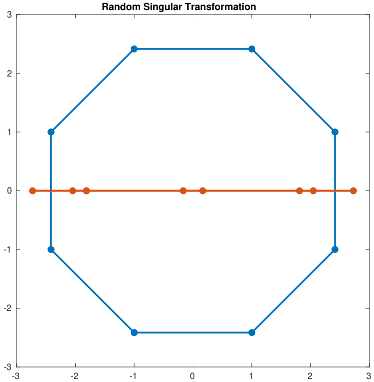



Can this transformation be undone?

Yes! `det(M) = 0.1381`

# Random non-invertible Transformation

$$M = \begin{bmatrix} 0.9884 & 0.3409 \\ 0.0000 & 0.0000 \end{bmatrix}$$



# Affine transformations

These are mappings of the form

$$\boldsymbol{T}(\vec{v}) = \boldsymbol{A}\vec{v} + \vec{b}$$

i.e. **affine transformations** are composed of a linear transformation ($\boldsymbol{A}\vec{v}$) then shifted in the direction $\vec{b}$

# Affine transformations

These are mappings of the form

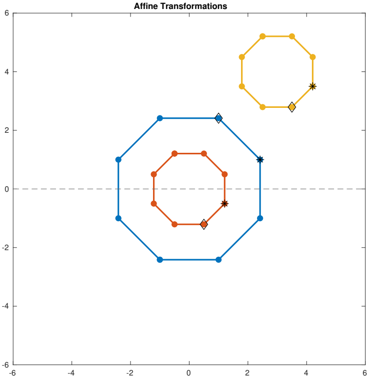
$$\boldsymbol{T}(\vec{v}) = \boldsymbol{A}\vec{v} + \vec{b}$$

i.e. **affine transformations** are composed of a linear transformation ($\boldsymbol{A}\vec{v}$) then shifted in the direction $\vec{b}$

- Affine transformations preserve collinearity and ratios of distances.
- Translations, dilations, contractions,reflections and rotations are all examples of affine transformations.

# Affine transformations

$$\boldsymbol{T} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



# Affine transformations and fractals

Consider **four** different linear transformations on points $\vec{v} = (x, y)$ starting at $(0, 0)$ and **one** linear transformation performed randomly with different probabilities

- 85% of the time:

$$\boldsymbol{T}_1 = \boldsymbol{A}_1 \vec{v} + \vec{b}_1 = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \vec{v} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

- 7% of the time:

$$\boldsymbol{T}_2 = \boldsymbol{A}_2 \vec{v} + \vec{b}_2 = \begin{bmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \vec{v} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

- 7% of the time:

$$\boldsymbol{T}_3 = \boldsymbol{A}_3 \vec{v} + \vec{b}_3 = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \vec{v} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}$$

- 1% of the time:

$$\boldsymbol{T}_4 = \boldsymbol{A}_4 \vec{v} = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \vec{v}$$

# Exercise: Affine transformations and fractals - Implementation notes

- Use `randsample(4,1,true,[0.85 0.07 0.07 0.01])` to generate random integers with weights
- Starting with the origin apply a transformation based on the outcome from `randsample`, (a `switch` statement may be useful here).
- plot each point after applying the transformation - use `drawnow` to visualize the points as they are computed.

# Affine transformations and fractals