Numerical Integration

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A jog down Calc I/II lane

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- In some practical cases, we do not have an analytical representation of f but we still want to approximate the integral
- Numerical integration techniques are necessary to approximate the integral

Approximating $\int_a^b f(x)\,dx$ (basic idea)

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- More sophisticated methods use adaptive widths of subinterval depending on the behavior of the function
- As the number of sub-intervals increases, we obtain a more accurate approximation of the area under the curve

Desmos demo

Approximating $\int_a^b f(x)\,dx$ (Implementation)

Rectangles

1. Divide $[a, b]$ so that

$$\boxed{a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b, \quad k = 1, 2, \ldots, n}$$

with $x_k = a + (k-1)\Delta x$

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2. On each sub-interval $[x_k, x_{k+1}]$ select a *sample point*, $x_k^* \in [x_k, x_{k+1}]$

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2. On each sub-interval $[x_k, x_{k+1}]$ select a *sample point*, $x_k^* \in [x_k, x_{k+1}]$
3. Define the height of each sub-rectangle as $f(x_k^*)$ so that the area of each sub-rectangle is

$$\boxed{f(x^*)\Delta x}$$

4. Summing up for the n sub-intervals

$$\boxed{\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(x_k^*)\Delta x}$$

Approximating $\int_a^b f(x)\,dx$ (Implementation)

Trapeziods

1. Each trapezoid has a base of $[x_k, x_{k+1}]$ with parallel sides of length $f(x_k)$ and $f(x_{k+1})$

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Approximating $\int_a^b f(x)\,dx$ (Implementation)

Trapezoids

$$\begin{aligned}
\int_a^b f(x)\,dx &\approx \frac{\Delta x}{2}\sum_{k=1}^{n}\left(f(x_k)+f(x_{k+1})\right)\\
&= \frac{\Delta x}{2}\left([f(x_1)+f(x_2)]+[(f(x_2)+f(x_3)]+\cdots+[f(x_{n-1})+f(x_n)]+[f(x_n)+f(x_{n+1})]\right)\\
&= \frac{\Delta x}{2}\left(f(x_1)+2f(x_2)+\cdots+2f(x_n)+f(x_{n+1})\right)
\end{aligned}$$

$$\boxed{\int_a^b f(x)\,dx = \frac{\Delta x}{2}\left(f(x_1)+2\sum_{k=2}^{n} f(x_k)+f(x_{n+1})\right)}$$

Error Analysis (MA 428)

Theorem

Assuming $\max_{a \le x \le b} |f''(x)| \le M$. *Then the midpoint method has error*

$$\boxed{\frac{M(b-a)(\Delta x)^2}{24}}$$

and the Trapezoidal method has error

$$\boxed{\frac{M(b-a)(\Delta x)^2}{12}}$$

Simpson's Method for approximating $\int_a^b f(x)\,dx$

Approximate the area under the function using second order curves

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1. Divide $[a, b]$ into n sub-intervals of width $\Delta x = \frac{b-a}{n}$, where n is **even**.

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Approximate the area under the function using second order curves

1. Divide $[a, b]$ into n sub-intervals of width $\Delta x = \frac{b-a}{n}$, where n is **even**.
2. On **each pair** of sub-intervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ $(k = 2, \cdots, n)$ approximate the area under the curve with a quadratic function passing through the points:

$$\boxed{\big(x_{k-1}, f(x_{k-1})\big), \big(x_k, f(x_k)\big) \text{ and } \big(x_{k+1}, f(x_{k+1})\big)}$$

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3. The area under each parabola on $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ is

$$\boxed{\frac{\Delta x}{3}\left(f(x_{k-1}) + 4f(x_k) + f(x_{k+1})\right)}$$

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$$\frac{\Delta x}{3}\left(f(x_{k-1}) + 4f(x_k) + f(x_{k+1})\right)$$

4. Summing up over all sub-intervals

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(f_1 + 4f_2 + 2f_3 + 4f_4 + \cdots + 2f_{n-1} + 4f_n + f_{n+1}\right)$$

Error Analysis (MA 428)

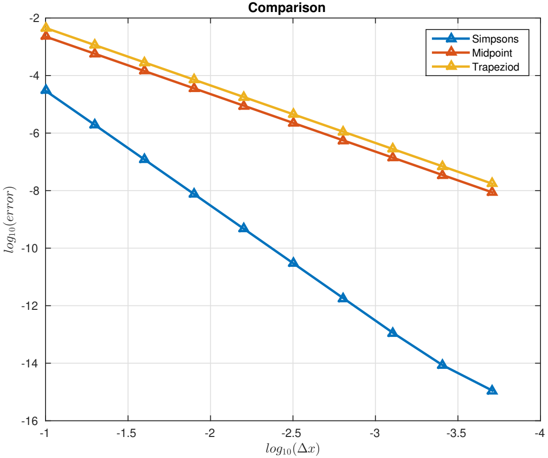
Theorem

Assuming $\max_{a\le x\le b} |f^{(4)}(x)| \le M$. *Then the Simpson's method has error*

$$\boxed{\frac{M(b-a)(\Delta x)^4}{180}}$$

- *Composite* Simpson's method has a convergence rate of $\mathcal{O}(\Delta x)^4$ compared to Midpoint and Trapezoidal that are $\mathcal{O}(\Delta x)^2$.

Error comparison



- Midpoint and Trapezoidal methods are second order in Δx i.e. $\mathcal{O}((\Delta x)^2)$
- Simpsons method is fourth order in Δx i.e. $\mathcal{O}((\Delta x)^4)$

Generalized formula

Higher order methods of the form

$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} f(x_i)w_i$$

- These methods can be extended to $2D$ and $3D$ integrals (see MA 428).