251 Midterm 1 preview: Fall, 2007

Things to keep in mind as you take this practice test:

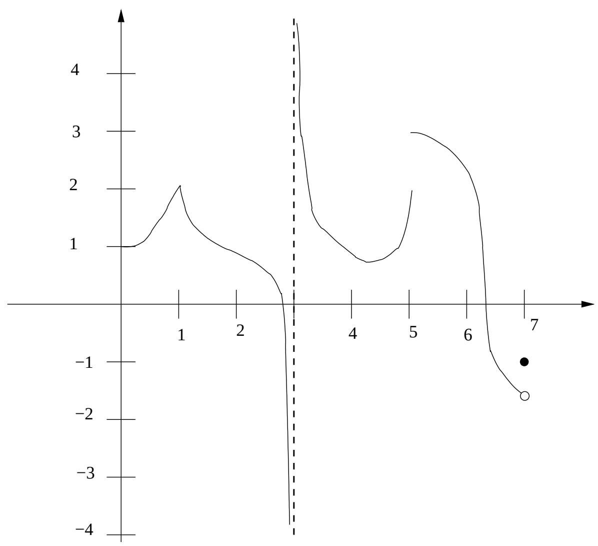
- The real test will not be this long. It will probably have around 8 problems.
- You should know/memorize/write down rules for exponents, values of sin, cos and tan at 0, $\pi/6$, $\pi/4$, $\pi/3$ and $\pi/2$ (as well as how to use these angles to move around the unit circle), basic methods of algebra (like foiling, simplifying etc), how to use e^x and $\ln(x)$, how to solve various equations for x.
- You should know/memorize the short cut derivitave rules for all powers of x (i.e. $\frac{d}{dx}x^n$), e^x, the sum rule, constant rule, the product rule, and the quotient rule. You can use the rules at any time unless the problem says otherwise.
- You should know that the derivative is used to find slopes of tangent lines and/or instantaneous velocity.
- You should know how to find limits using tables of numbers, graphs, and/or algebra.
- Since this test is for practice you should think about doing variations of some of the problems, especially the ones that you find difficult.
- Everything should be done algebraicly unless explicitly stated otherwise, or where it is not applicable, like a problem involving only the picture of a graph or a table of numbers.
- Problem that ask you to "show that ... is true" or "justify your answer" or something like that, mean this: Give a clear and complete sequence of steps, each of which is obvious from an algebraic point of view, or which quotes one of our main results, like the Squeeze Theorem, or the Intermediate Value Theorem.

1. A calculator was used to produce the following table of numbers for $f(x) = \frac{1-\cos(x)}{x^2}$. Using the table, what is $\lim_{x\to 0} f(x)$?

x	$-.1$	$-.01$	0	.01	.1
$f(x)$	.49958	.499999999	ERROR	.4999999	.49958

Solution. The limit appears to be .5.
Justification/explanation. We are given numbers with x-values close to 0. In particular, $-.1$, $-.01$, .01, and .1. The closer to zero that x is, the closer y gets to .5.

2. Using the graph below of $f(x)$, find the following limits (or indicate that the limit does not exist):



(a) $\lim_{x\to 1} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3^-} f(x)$

(d) $\lim_{x\to 3} f(x)$

(e) $\lim_{x\to 5^+} f(x)$

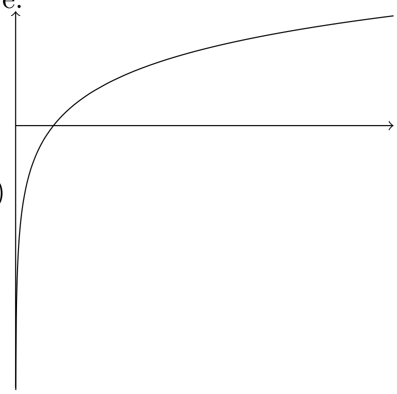
(f) $\lim_{x\to 5^-} f(x)$

(g) $\lim_{x\to 7^-} f(x)$

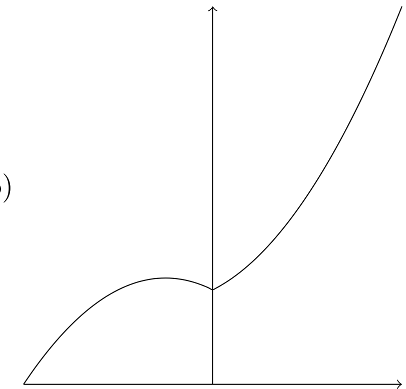
Solution. (a) The limit equals 2 because the y-values get infinitely close to 2 as the x-values get infinitely close to 1.
(b) The limit is ∞ because the y-values get infinitely large as the x-values get infinitely close to 3 from the right (i.e. with $x > 3$).
(c) The limit is $-\infty$ because the y-values get infinitely far down the y-axis as the x-values get infinitely close to 3 from the left (i.e. with $x < 3$).
(d) The limit DNE because there is no single L that we can say all the y-values get close to as x gets close to 3.
(e) The limit is 3 because the y-values get infinitely close to 3 as the x-values get infinitely close to 5 from the right (i.e. with $x > 5$).
(f) The limit is 2 because the y-values get infinitely close to 2 as the x-values get infinitely close to 5 from the left (i.e. with $x < 5$).
(g) The limit is -2 because the y-values get infinitely close to -2 as the x-values get infinitely close to 7 from the left (i.e. with $x < 7$).

3. Graphs (a)–(d) below are the graphs of four functions. Graphs I–IV are the graphs of their derivatives. For each function (a)–(d), indicate which of the graphs I-IV could be its derivative.

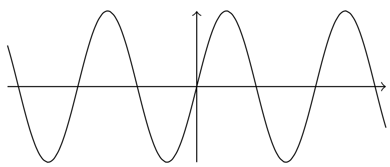
(a)



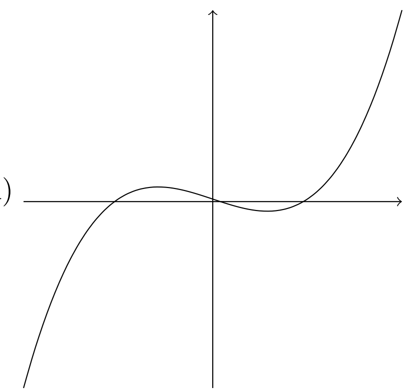
(b)



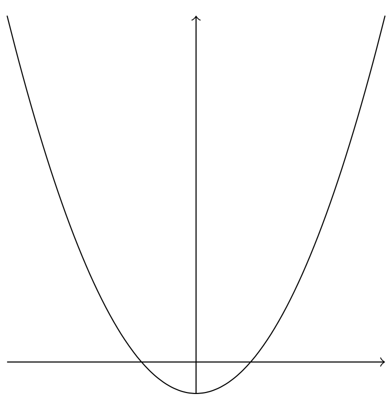
(c)



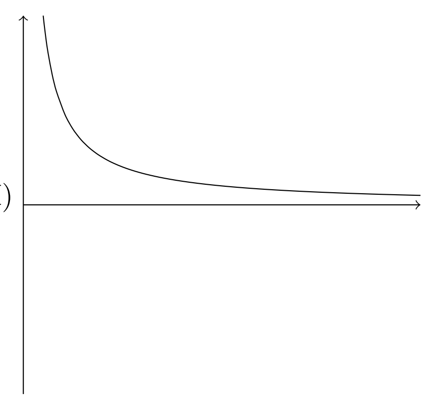
(d)



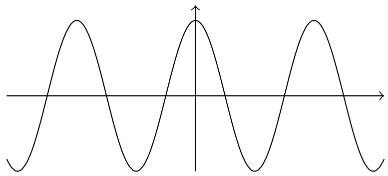
(I)



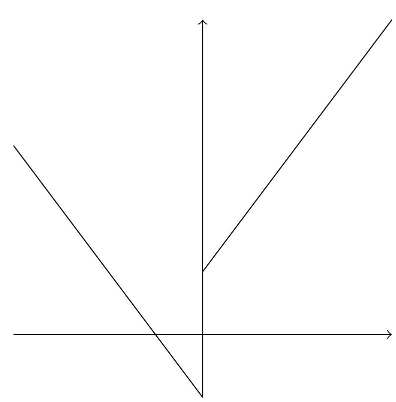
(II)



(III)



(IV)



Solution. The derivative of (a) is (II). The derivative of (b) is (IV). The derivative of (c) is (III). The derivative of (d) is (I).

In each case you can verify the solution by constantly checking things like "Here the slope of (a) is positive, and here the y-values of (II) are positive" where "here" refers to the same range of x-values.

4. Find $\lim_{x\to 0} x^4 \cos(2/x)$. (Hint: use the squeeze theorem and the fact that $-1 \leq \cos(\ ) \leq 1$.)

Solution. We will apply the squeez theorem with $f(x) \leq g(x) \leq h(x)$. Let $g(x) = x^4 \cos(2/x)$. To find f and h we replace $\cos(\ )$ with -1 and 1:

$$x^4(-1) \leq x^4 \cos(2/x) \leq x^4(1).$$

Let $f(x) = -x^4$ and $h(x) = x^4$. Then we have

$$\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0.$$

Therefore, by the Squeeze Theorem we have that $\lim_{x\to 0} g(x) = 0$ as well.

5. Working only with integers, and without using your calculator, show that the equation $x^2 - 101 = 0$ has a solution, and give an interval $[a, b]$ which contains this solution.

One way to think about this is that you are showing that $\sqrt{101}$ really exists, and you are giving an integer a which is too low, and another integer b which is too high. Here's what you should actually do.

Show me an integer a which makes the function negative. Show me an integer b which makes the function positive. Quote the IMV (Intermediate Value Theorem) to give your conclusion.

Solution. Let $f(x) = x^2 - 101$. Note that $f(10) = 100 - 101 = -1$, which is negative. Note that $f(11) = 121 - 101 = 20$, which is positive. Therefore, by the IMV, there is a solution of $f(x) = 0$ for some x in $[10, 11]$.

6. Find the following (simple) limits

(a) $\lim_{x\to 0} e^x \tan(x)$.

(b) $\lim_{x\to 3} \dfrac{x^2 - x + 12}{x + 3}$.

Solution. (a) We plug $x = 0$ in to get $e^0 \tan(0)$. Since everything is defined here, this gives us the limit. It simplifies to 0.

(b) We plug $x = 3$ in to get $\frac{3^2-3+12}{3+3} = \frac{18}{6}$.

7. Suppose a falling rock has position given by the following formula:

$$p(t) = -4.9t^2 + 13t + 10.$$

Find a formula for the instantaneous velocity of the rock.

Solution. We use the short cut rules to find the derivative:

$$p'(t) = -9.8t + 13$$

8. Find a such that the following function is continuous at $x = 0$ and justify your finding.

$$f(x) = \begin{cases} \sqrt{x} + a & \text{if } x \geq 0 \\ \sqrt{-x} + \cos(x) & \text{if } x < 0 \end{cases}$$

Solution. We need $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$. The limit from the right uses the formula $\sqrt{x}+a$ and the limit from the left uses the formula $\sqrt{-x}+\cos(x)$.

So we need

$$\lim_{x\to 0^+} \sqrt{x}+a = \lim_{x\to 0^-} \sqrt{-x}+\cos(x)$$

Finally, we just plug in $x=0$ to find these limits

$$\sqrt{0}+a = \sqrt{0}+\cos(0)$$

from which we see that $a=1$.

9. Solve the following equations for x. (This does not mean find a decimal solution. Instead, give an algebraic solution of x. For instance, you would write $x=\sqrt{2}$ not $x=1.414$.)

   (a) $e^x = 7$
   (b) $\ln(x-1) = 17$
   (c) $\pi x = 500/x^2$
   (d) $x^2+7x-13=0$
   (e) $\sin(x) = -\sqrt{3}/2$ (find only one solution of x)
   (f) $\sin(x^2) = \frac{1}{2}$ (find only one solution of x)

   Solution. (a) We take ln of both sides to get $x=\ln(7)$.

   (b) We exponentiate both sides to get $x-1=e^{17}$, so $x=e^{17}+1$.

   (c) We multiply both sides by x^2 to get $\pi x^3 = 500$, then divide by π to get $x^3 = 500/\pi$, and then take the cube root $x=\sqrt[3]{500/\pi}$.

   (d) We apply the quadratic formula

$$x = \frac{-7\pm\sqrt{7^2+4\cdot 1\cdot 13}}{2} = \frac{-7\pm\sqrt{101}}{2}$$

   (e) We know that $\sin(\pi/3) = \sqrt{3}/2$. To make a negative output we can take any angle below the x-axis. So, one solution is given by $x=-\pi/3$. Another solution is given by $x=-2\pi/3$. All other solutions equal one of these plus some multiple of 2π.

   (f) We know that $\sin(\pi/6)=1/2$. Thus, we should have $x^2=\pi/6$ and $x=\sqrt{\pi/6}$.

10. Find the following derivatives:

   (a) $\dfrac{d}{dx}10x^9$
   (b) $\dfrac{d}{dx}[-\sqrt{x}+10x^9]$
   (c) $\dfrac{d}{dx}[\dfrac{1}{x}+\pi x^{3.14159}]$
   (d) $\dfrac{d}{dx}[x+x^2+x^3+x^4+x^5+e^x]$.
   (e) $\dfrac{d}{dx}[\sqrt{x}e^x]$
   (f) $\dfrac{d}{dx}\dfrac{10x^{7.3}-\sqrt{x}}{10e^x+x}$

Solution. (a) $9 \cdot 10x^8$.

(b) $-\frac{1}{2\sqrt{x}} + 9 \cdot 10x^8$

(c) $-\frac{1}{x^2} + \pi \cdot 3.14159x^{2.14159}$

(d) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + e^x$

(e) $\frac{e^x}{2\sqrt{x}} + \sqrt{x}e^x$

(f) $\frac{(7.3 \cdot 10x^{6.3} - \frac{1}{2\sqrt{x}})(10e^x + x) - (10x^{7.3} - \sqrt{x})(10e^x + 1)}{(10e^x + x)^2}$

11. Simplify $\dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$ as much as possible (note: I am *not* asking you to plug in $h = 0$ or to find the limit $\lim\limits_{h \to 0}$ or to finish taking the derivative, etc. Just simplify.)

Solution.

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x}{x(x+h)} - \frac{(x+h)}{x(x+h)}}{h} = \frac{x - (x+h)}{x(x+h)} \cdot \frac{1}{h} = \frac{-h}{x(x+h)} \cdot \frac{1}{h} = \frac{-1}{x(x+h)}$$

12. Using the definition, find the derivatives of the following functions, at the indicated point:

(a) $f(x) = -5x^2 + 2x$, find $f'(3)$.

(b) $f(x) = \frac{1}{x+1}$, find $f'(5)$.

(c) $f(x) = \sqrt{2x+1}$, find $f'(2)$.

Solution. (a) $f'(3) = \lim\limits_{h \to 0} \dfrac{-5(3+h)^2 + 2(3+h) - (-5 \cdot 3^2 + 2 \cdot 3)}{h}$
$= \lim\limits_{h \to 0} \dfrac{-5(9 + 6h + h^2) + 6 + 2h + 45 - 6}{h} = \lim\limits_{h \to 0} \dfrac{-45 - 30h - 5h^2 + 6 + 2h + 45 - 6}{h} =$
$\lim\limits_{h \to 0} \dfrac{-30h - 5h^2 + 2h}{h} = \lim\limits_{h \to 0} -30 - 5h^2 + 2 = -28$

(b) $f'(5) = \lim\limits_{h \to 0} \dfrac{\frac{1}{5+h+1} - \frac{1}{5+1}}{h} = \lim\limits_{h \to 0} \dfrac{\frac{6}{6(h+6)} - \frac{(h+6)}{6(h+6)}}{h} = \lim\limits_{h \to 0} \dfrac{6 - (h+6)}{6(h+6)} \dfrac{1}{h} = \lim\limits_{h \to 0} \dfrac{-h}{6(h+6)} \dfrac{1}{h} =$
$\lim\limits_{h \to 0} \dfrac{-1}{6(h+6)} = \dfrac{-1}{6(0+6)} = \dfrac{-1}{6^2}$

(c) $f'(4) = \lim\limits_{h \to 0} \dfrac{\sqrt{2(2+h)+1} - \sqrt{4+1}}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{2h+5} - \sqrt{5}}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{2h+5} - \sqrt{5}}{h} \dfrac{\sqrt{2h+5} + \sqrt{5}}{\sqrt{2h+5} + \sqrt{5}} =$
$\lim\limits_{h \to 0} \dfrac{(2h+5) - 5}{h(\sqrt{2h+5} + \sqrt{5})} = \lim\limits_{h \to 0} \dfrac{2h}{h(\sqrt{2h+5} + \sqrt{5})} = \lim\limits_{h \to 0} \dfrac{2}{\sqrt{2h+5} + \sqrt{5}} = \dfrac{2}{\sqrt{0+5} + \sqrt{5}} =$
$\dfrac{2}{2\sqrt{5}} = \dfrac{1}{\sqrt{5}}$

13. Let $f(x) = x^3 - 2x^2 - 3x + 10$.

(a) Find $f'(x)$.

(b) Find the equation of the tangent line at $x = 1$.

Solution. (a) $f'(x) = 3x^2 - 4x - 3$

(b) The slope at $x = 1$ equals $3 \cdot 1^2 - 4 \cdot 1 - 3 = -4$. The x_0-value is given by 1. The corresponding y_0-value is given by $f(1) = 1^3 - 2 \cdot 1^2 - 3 \cdot 1 + 10 = 1 - 2 - 3 + 10 = 6$. Thus, the equation of the tangent line is

$$y = -4(x - 1) + 6$$

14. Find the following limit, justify your answer by using a clear and complete sequnce of steps.

$$\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}$$

Solution. We divide the top and the bottom by x^2, since that is the biggest power of x that is on the bottom.

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}(x+2)}{\frac{1}{x^2}(9x^2+1)} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{2}{x^2}}{9+\frac{1}{x^2}} = \frac{0+0}{9+0}$$

where every "0" was from the fact that $\lim_{x\to\infty} 1/x^r = 0$. Thus, the total limit is 0.