Math 251, Spring 2005: Exam #2 Preview Problems

1. Using the definition of derivative find the derivative of the following functions:

   (a) $f(x) = e^x$. (Use the following $\lim_{h\to 0} \frac{e^h - 1}{h} = 1, \quad e^{x+h} = e^x e^h$.)

   (b) $f(x) = \frac{1}{x^2}$

Solution. (a) $f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = \lim_{h\to 0} \frac{e^x}{1} \frac{e^h - 1}{h} = e^x \cdot 1$

(b)

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \left(\frac{x^2}{(x+h)^2 x^2} - \frac{(x+h)^2}{(x+h)^2 x^2} \right) \frac{1}{h} = \lim_{h\to 0} \frac{x^2 - (x+h)^2}{(x+h)^2 x^2} \frac{1}{h}$$
$$= \lim_{h\to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{(x+h)^2 x^2} \frac{1}{h} = \lim_{h\to 0} \frac{-2xh + h^2}{(x+h)^2 x^2} \frac{1}{h} = \lim_{h\to 0} \frac{-2x + h}{(x+h)^2 x^2} = \frac{-2x + 0}{(x+0)^2 x^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}$$

□

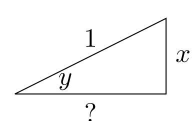
2. Using implicit derivatives and the derivative of sin find/verify the shortcut derivative rule for $\sin^{-1}(x)$. (Hint: start with $y = \sin^{-1}(x)$, (a) solve this for x, (b) take the implicit derivative, solve for y', (c) to rewrite the formula so that you don't have y in the formula any more use a right triangle, make one of the angles y, and label the other two sides using the equation that you got as an answer to part (a).)

Solution. (a) We start with $y = \sin^{-1}(x)$ and rewrite this to get $\sin(y) = x$.

(b) We take implicit derivatives of $\sin(y) = x$ to get

$$\cos(y) y' = 1$$
$$y' = \frac{1}{\cos(y)}$$

(c) We draw a triangle using $\sin(y) = x = \frac{x}{1}$



We solve for $? = \sqrt{1 - x^2}$ and so $\cos(y) = \sqrt{1 - x^2}$ and so

$$y' = \frac{1}{\sqrt{1 - x^2}}$$

□

3. Find the derivative of the following functions:

   (a) $y = (x^4 - 3x^2 + 5)^3$

   (b) $y = \ln(\sec(2x + 1)) - \frac{1}{2} \sin^2(x)$

   (c) $y = e^{-t}(t^2 - 2t + 2)$

(d) $y = 2x\sqrt{x^2+1}$.

(e) $y = (\cos(x))^x$

Solution. (a) $5(x^4 - 3x^2 + 5)^2(4x^3 - 6x)$.

(b) $\dfrac{1}{\sec(2x+1)}\sec(2x+1)\tan(2x+1)\cdot 2 - \sin(x)\cos(x) = 2\tan(2x+1) - \sin(x)\cos(x)$.

(c) $-e^{-t}(t^2 - 2t + 2) + e^{-t}(2t - 2)$.

(d) $2\sqrt{x^2+1} + 2x\frac{1}{2}(x^2+1)^{-1/2}2x = 2\sqrt{x^2+1} + \dfrac{x^2}{\sqrt{x^2+1}}$

(e) $(\cos(x))^x\left(\ln(\cos(x)) - \dfrac{x\sin(x)}{\cos(x)}\right)$

□

4. Suppose that $y = y(x)$ satisfies the equation $2xy - \ln y = 4$ and the condition.

   (a) Using implicit derivatives, calculate y' at the point $(2, 1)$.

   (b) Using your derivative from (a), find the equation of the tangent $L(x)$ at $(2, 1)$.

   (c) Using linear approximation, get an approximate value of $y(2.1)$.

Solution. (a) Implicit derivatives give

$$\begin{aligned} 2(1\cdot y + xy') - \frac{1}{y}\cdot y' &= 0 \\ 2y + 2xy' - \frac{1}{y}\cdot y' &= 0 \\ (2x - \frac{1}{y})y' &= -2x \\ y' &= \frac{-2y}{2x - 1/y} \end{aligned}$$

At the point $(2, 1)$ this gives

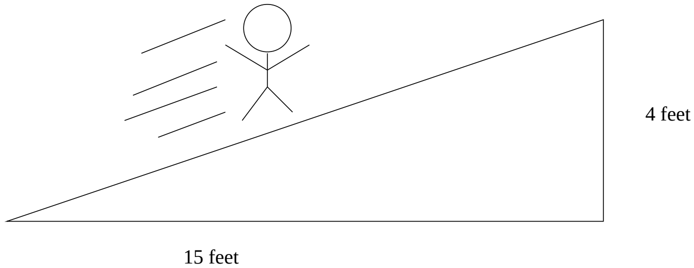
$$y' = -2/3$$

(b) $L(x) = -\frac{2}{3}(x-2) + 1$.

(c) $y(2.1) \approx L(2.1) = -\frac{2}{3}(2.1 - 2) + 1 = 1 - .06666$

□

5. A waterskier goes over a jump at a speed of 30 ft/s (this is her speed going along the diagonal). Find the speed at which she is rising (i.e. find the vertial speed) as she leaves the ramp.



Solution. Let x and y be the water skier's horizontal and vertical distances from the start. Then similar triangles gives

$$\frac{x}{y} = \frac{15}{4} \Rightarrow x = \frac{15}{4}y$$

From the Pythagorean theorem we have $c^2 = x^2 + y^2$. Combining this with the previous formula gives

$$c^2 = (\frac{15}{4}y)^2 + y^2$$
$$c^2 = (\frac{15^2}{4^2} + 1)y^2$$
$$c = \sqrt{\frac{15^2}{4^2} + 1}y$$

Now we take the time derivative of this equation

$$\frac{d}{dt}c = \frac{d}{dt}\sqrt{\frac{15^2}{4^2} + 1}y$$

We are given that $\frac{d}{dt}c = 30$ so we have

$$y = \frac{30}{\sqrt{\frac{15^2}{4^2} + 1}} = \frac{120}{\sqrt{241}}$$

□

6. If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \frac{t}{40}\right)^2 \qquad 0 \leq t \leq 40.$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At which of these times is the water flowing the fastest? Which time is the water flowing the slowest?

Solution. We take the time derivative $\frac{d}{dt}$ to get

$$\frac{d}{dt}V = 5000 \cdot 2\left(1 - \frac{t}{40}\right) \cdot (0 - \frac{1}{40})$$

At $t = 5$ this gives $-\frac{875}{4}$. At $t = 10$ this gives $-\frac{375}{2}$. At $t = 20$ this gives -125. At $t = 40$ this gives 0.

The water is flowing fastest at $t = 5$ and slowest at $t = 40$. □

7. Find the absolute minimum and maximum of the function $f(t) = t + \cot(t/2)$, on the interval $[\pi/4, 7\pi/4]$.

Solution. We start with the derivative $f'(t) = 1 - \csc^2(t/2) \cdot \frac{1}{2}$. We solve for $f'(t) = 0$

$$\begin{aligned}
1 - \frac{1}{2}\csc^2(t/2) &= 0 \\
2 &= \csc^2(t/2) \\
\pm\sqrt{2} &= \csc(t/2) \\
\pm 1/\sqrt{2} &= \sin(t/2) \\
t/2 &= \pm\pi/4, \pm 3\pi/4, \pm 5\pi/4, \pm 7\pi/4, \ldots \\
t &= \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \pm 7\pi/2, \ldots
\end{aligned}$$

We take only those values of t in the interval $[\pi/4, 7\pi/4]$

$$t = \pi/2, 3\pi/2$$

Now we also consider when $f'(t)$ is undefined: csc is undefined when sin equals 0. This is at $\theta = 0$, π, 2π,.... Here we have $\theta = t/2$ so $f'(t)$ is undefined at $t = 0$, 2π, 4π, etc., none of which are contained in the interval $[\pi/4, 7\pi/4]$.

Now we compare y-values

$$\begin{aligned}
f(\pi/4) &= \frac{\pi}{4} + \cot(\pi/8) \approx 3.2 \\
f(\pi/2) &= \frac{\pi}{2} + 1 \approx 2.6 \\
f(3\pi/2) &= \frac{3\pi}{2} - 1 \approx 3.7 \\
f(7\pi/4) &= \frac{7\pi}{4} - \cot(\pi/8) \approx 3.1
\end{aligned}$$

From this we see that the Absolute max is at $3\pi/2$ and the absolute min is at $\pi/2$.

□

8. Let $f(x) = x^2 \ln(x)$.
   (a) Find the critical points.
   (b) Find the intervals of increase and decrease and identify each critical point as a local maximum, minimum or neither.
   (c) Find the intervals of concavity.

Solution. (a) $f'(x) = 2x\ln(x) + \frac{x^2}{x} = 2x\ln(x) + x$. Set this equal to 0 and solve

$$\begin{aligned}
2x\ln(x) + x &= 0 \\
x(2\ln(x) + 1) &= 0 \\
x &= 0 \text{ or } 2\ln(x) + 1 = 0 \\
x &= 0 \text{ or } 2\ln(x) = -1 \\
x &= 0 \text{ or } \ln(x) = -1/2 \\
x &= 0 \text{ or } x = e^{-1/2}
\end{aligned}$$

We note also that $f'(x)$ is undefined at $x = 0$, but so is $f(x)$. Therefore, $x = 0$ is not a critical number. The only critical number is $x = e^{-1/2} = 1/\sqrt{e}$.

(b) We have two intervals: $(0, 1/\sqrt{e})$ and $(1/\sqrt{e}, \infty)$. We test one x-value from each of these intervals in $f'(x)$:

$$f'(.1) = .1(2\ln(.1) + 1) \approx .1(2(-2.3) + 1) < 0$$
$$f'(e) = e(2\ln(e) + 1) = e(2 + 1) > 0$$

From this, combined with the first derivative test, we see that

Interval of increase:	$(1/\sqrt{e}, \infty)$
Interval of decrease:	$(0, 1/\sqrt{e})$
Local min:	$x = 1/\sqrt{e}$

(c) We take the second derivative

$$f''(x) = 2\ln(x) + 2x\frac{1}{x} + 1 = 2\ln(x) + 3$$

We solve for when this equals 0

$$\begin{aligned} 2\ln(x) + 3 &= 0 \\ \ln(x) &= -3/2 \\ x &= e^{-3/2} \end{aligned}$$

This gives us two intervals $(0, e^{-3/2})$, $(e^{-3/2}, \infty)$.
We test one x-value in each of these intervals

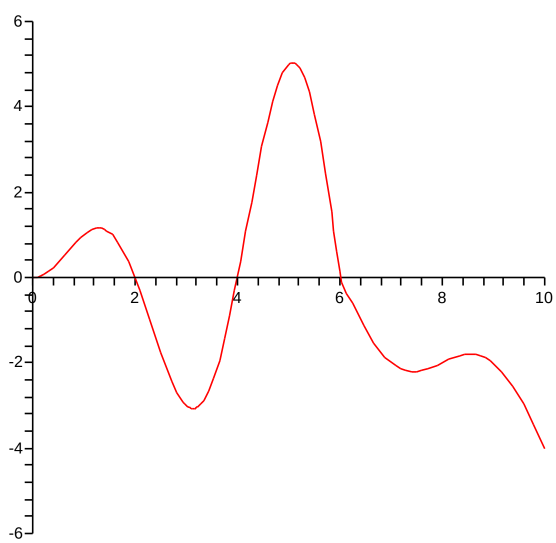
$$f''(.1) = 2\ln(.1) + 3 < 0$$
$$f''(e) = 2\ln(e) + 3 > 0$$

From this we see that

Concave up:	$(e^{-3/2}, \infty)$
Concave down:	$(0, e^{-3/2})$

□

9. The graph of the first derivative $f'(x)$ is shown below (the function doesn't exist to the left of zero, and to the right of 10 it just keeps going in the same direction that is shown).



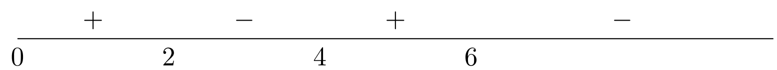
(a) Find the critical points of $f(x)$.

(b) Find the intervals of increase and decrease of $f(x)$ and identify each critical point as a local maximum, minimum or neither.

(c) Find the intervals of concavity of $f(x)$.

Solution. (a) We see that $f'(x) = 0$ at $x \approx 0, 2, 4, 6$.

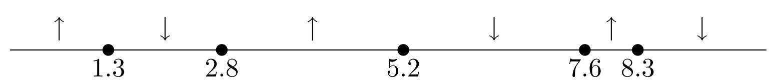
(b) From the graph we see the following about when $f'(x)$ is positive and negative



From which we conclude that

Intervals of increase:	$(0, 2) \cup (4, 6)$
Intervals of decrease:	$(2, 4) \cup (6, \infty)$
Local max's:	$x = 2, 6$
Local min's:	$x = 4$

(c) For the concavity of $f(x)$ we must look at when $f''(x)$ is positive or negative, which means the same thing as looking at when $f'(x)$ is increasing or decreasing. Judging from the graph we have that $f'(x) = 0$ at $x \approx 1.3, 2.8, 5.2, 7.6, 8.3$. Between this points we have that $f'(x)$ is increasing or decreasing as pictured



From this we see that the intervals of concavity are

$$\text{concave up: } (0, 1.3) \cup (2.8, 5.2) \cup (7.6, 8.3), \qquad \text{concave down: } (1.3, 2.8) \cup (5.2, 7.6) \cup (8.3, \infty)$$

□