Midterm 3 preview solutions

Math 251, Calculus I, Fall 2007

Directions. Practice these problems, and variations on them.

Practice doing each problem algebraically, as much as you can. Algebra, once you're good at it, is faster than using a calculator, gives more accurate results, is what will be required in more of your math and science classes in the future, and makes you smarter and understand the math better.

Practice showing your work; work which justifies your answer.

Memorize as many derivative and anti-derivative formulas as you can: this will make you much faster on the test and prevent mistakes due to mis-copying.

1. Calculate y'

   (a) $y = \dfrac{3x-2}{\sqrt{2x+1}}$

   (b) $y = \ln(\csc(5x))$

   (c) $x^2\cos(y) + \sin(2y) = xy$

   (d) $y = (\cos(x))^x$

   Solution:

   (a) $y' = \dfrac{3\sqrt{2x+1} - (3x-2)\frac{1}{2}(2x+1)^{-1/2}\cdot 2}{2x+1}$

   (b) $y' = \dfrac{1}{\csc(5x)}(-\csc(5x)\cot(5x))5$

   (c) $y' = -\dfrac{2x\cos(y) - y}{-x^2\sin(y) + 2\cos(2y) - x}$

   (d) $y' = (\cos(x))^x\left(\ln(\cos(x)) - \dfrac{x\sin(x)}{\cos(x)}\right)$

2. Let $f(x) = 1/x$ and let $g(x)$ be defined as

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{x} + 1 & \text{if } x < 0 \end{cases}$$

   Show that $f'(x) = g'(x)$ for all x ("all" means all x where the functions are defined). Can we conclude from the corollary to the mean value theorem that $f(x) = g(x) + C$ for all x? Why or why not?

   Solution: $f'(x) = -\dfrac{1}{x^2}$ and

$$g'(x) = \begin{cases} -\dfrac{1}{x^2} & \text{if } x > 0 \\ -\dfrac{1}{x^2} + 0 & \text{if } x < 0 \end{cases}$$

   This shows that $f'(x) = g'(x)$ for all x.

   However, it is not the case that $f(x) = g(x) + C$ since we would have $C = 0$ for $x > 0$ and $C = 1$ for $x > 0$. The reason the corollary to the mean value theorem fails is that $f(x)$ and $g(x)$ are not continuous for all x: these functions are not continuous at $x = 0$.

3. Find the limits

   (a) $\displaystyle\lim_{x\to\infty} \frac{\ln(\ln(x))}{x}$

   (b) $\displaystyle\lim_{x\to\infty} x\tan(1/x)$

(c) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}$

Solution:

(a) Use L'Hospital's Rule to get 0.

(b) Rewrite as a fraction and then use L'Hospital's Rule to get 1.

(c) Take ln( ), rewrite as a fraction and then use L'Hospital's Rule to get 6, raise as an exponent of e to get e^6.

4. For the function $f(x) = x^{5/3} - 5x^{2/3}$ find

(a) The x and y intercepts.

(b) The critical points.

(c) The intervals of increase and decrease.

(d) The local max/mins (be sure to show how you used the first/second derivative test).

(e) The vertical and horizontal asymptotes.

Solution:

(a) y-intercept: $y = 0$.
x-intercept: Solve $0 = x^{5/3} - 5x^{2/3}$ by factoring out $x^{2/3}$ to get $0 = x^{2/3}(x-5)$ and so $x = 0$ or $x = 5$.

(b) Take $f'(x)$, set it equal to 0 to get $x = 0, 2$.

(c) Decreasing from $(0, 2)$.
Increasing $(-\infty, 0) \cup (2, \infty)$.

(d) Local min at $x = 2$ since $f'(x)$ is $-$ to the left and $f'(x)$ is $+$ to the right.
Local max at $x = 0$ since $f'(x)$ is $+$ on the left and $-$ on the right.

(e) No HA or VA.

5. Turn the following limit into a definite integral on the interval $[1, 8]$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\cos(x_i)}{x_i}\Delta x$$

Solution: $\displaystyle\int_1^8 \frac{\cos(x)}{x}\,dx$

6. Give a Riemann sum, with $n = 5$ for the left hand, right hand and midpoint rules of the integral $\displaystyle\int_1^3 \cos^4(x)\,dx$

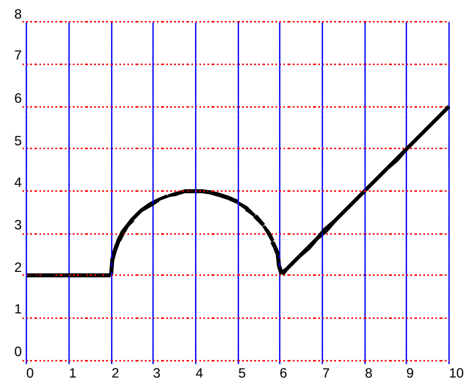
Solution: $\Delta x = 2/5 = .4$ so the rectangles have edges at $1, 1.4, 1.8, 2.2, 2.6, 3$.

LHR: $.4(\cos^4(1) + \cos^4(1.4) + \cos^4(1.8) + \cos^4(2.2) + \cos^4(2.6))$

RHR: $.4(\cos^4(1.4) + \cos^4(1.8) + \cos^4(2.2) + \cos^4(2.6) + \cos^4(3))$

MP: Here we plug in x-values half way between the other ones. The first is 1.2, and then they increase by .4 each time: $.4(\cos^4(1.2) + \cos^4(1.6) + \cos^4(2) + \cos^4(2.4) + \cos^4(2.8))$.

7. Below is a graph of a function $f(x)$ made of two straight lines and half of a circle. Find the indicated integrals



(a) $\int_0^2 f(x)\,dx$ (b) $\int_0^4 f(x)\,dx$ (c) $\int_2^8 f(x)\,dx$

Solution:

(a) 2^2

(b) $2^2 + 2^2 + \frac{1}{4}\pi 2^2$

(c) $2 \cdot 6 + \frac{1}{2}\pi 2^2 + \frac{1}{2} \cdot 2 \cdot 2$

8. Suppose that volume is given by $V = \pi r^2 h$ and surface area is given by $SA = 2\pi r^2 + 2\pi rh$.

   Find the maximum of V (as well as the r and h which give the maximum), given that the surface area equals 100.

   Solution: Solve $100 = 2\pi r^2 + 2\pi rh$ for $h = \dfrac{100 - 2\pi r^2}{2\pi r}$. Then $V = \pi r^2 \dfrac{100 - 2\pi r^2}{2\pi r} = 50r - \pi r^3$ whence $V' = 50 - 3\pi r^2$. Set this equal to zero and solve for $r = \sqrt{50/3\pi}$. Plug this in to get h and V.

9. Find the following integrals.

   (a) $\int_1^3 1 + 2x - 4x^3\,dx$.

   (b) $\int_0^5 2e^x + 4\cos(x)\,dx$.

   (c) $\int_0^{\pi/4} \sec^2(x)\,dx$.

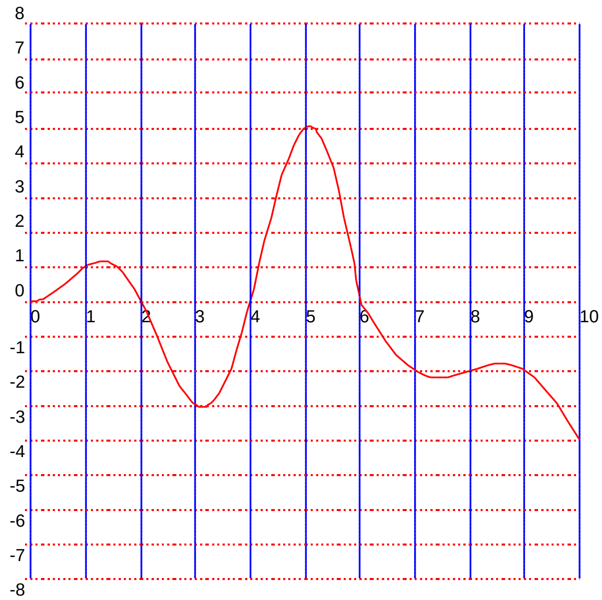
   Solution:

   (a) $x + x^2 - x^4\Big|_1^3 = -70$

   (b) $2e^x + 4\sin(x)\Big|_0^5 = 2e^5 + 4\sin(5) - 2$

   (c) $\tan(x)\Big|_0^{\pi/4} = 1$

10. Using the graph below, estimate the values of c that satisfy the conclusion of the mean value theorem on the interval $[0, 10]$



Solution: To my eyes the line through $(0,0)$ and $(10,-4)$ has the same slope as the tangent lines at the following values of c: 1.2, 3, 5, 7.2, 8.6.

11. Suppose that a mass bouncing on the end of a spring has velocity given by $v(t) = -10\cos(t)$, and that it has position $p = 20$ at $t = 0$. Find a formula for the position of the object as a function of t.

    Solution: $p(t) = 10\cos(t) + C$ and $p(0) = 20$ implies $C = 10$.

12. Use the Fundamental Theorem of Calculus part I to find the derivative of $f(x) = \int_0^{1/x} \tan^{-1}(t)\,dt$

    Solution: We apply the chain rule to get

$$\frac{d}{dx}\int_0^{1/x} \tan^{-1}(t)\,dt = \tan^{-1}(1/x)\cdot\frac{-1}{x^2}$$

13. Use the Fundamental Theorem of Calculus part II to find the following definite integrals

    (a) $\int_1^8 \sqrt[3]{x}\,dx$

    (b) $\int_0^{\pi/4} \sec(\theta)\tan(\theta)\,d\theta$

    (c) $\int_0^1 \frac{4}{t^2+1}\,dt$

    Solution:

    (a) $\left.\frac{3}{4}x^{4/3}\right|_1^8 = 6\cdot 8^{1/3} - \frac{3}{4}$

    (b) $\left.\sec(\theta)\right|_0^{\pi/4} = \sqrt{2} - 1$

    (c) $\left.4\tan^{-1}(t)\right|_0^1 = \pi.$