

## Workshop on finding roots

Most of the world around you can be described with equations, and every equation begs to be solved. Sometimes “solving” means doing it algebraically, but sometimes a numerical solution is good enough. This workshop describes two methods of finding a numerical solution of an equation.

Any equation is equivalent to one where one side is zero,

$$\text{stuff} = \text{stuff} \quad \longrightarrow \quad \text{left stuff} - \text{right stuff} = 0.$$

Solving an equation where one side is 0 is called **finding a root** of the equation. It turns out that there are certain techniques that are easiest to use when they are applied to an equation where one side is zero, so we will study how to find roots.

Our first method of finding roots is called the **Method of Bisection**. Here’s how it works. Using the Intermediate Value Theorem (IMV) you show that an equation has a solution on an interval  $[a, b]$ . Then you bisect this interval (split it in half), and use the IMV again to decide which half of the interval contains the solution. You split the new interval in half again, and so on, repeating until the final interval is small enough that it’s midpoint can be taken as the solution within a certain amount of inaccuracy. Throughout this workshop we will be working with an inaccuracy of .005. We consider the following to be equivalent statements about the accuracy, or inaccuracy, of a solution

“the solution is  $x = 2 \pm .005$ ”

“the solution  $x$  lies between 1.995 and 2.005”

“the solution is  $x = 2$  with an error  $\leq .005$ ”

“the solution is  $x = 2.00$  with an accuracy of .005”

Define the function

$$f(x) = x^4 - 18x^2 - 15. \tag{1}$$

Prove (using the intermediate value theorem) that the equation  $f(x) = 0$  has a solution  $x$  in the interval  $[4, 6]$ . Feel free to enter  $f(x)$  into the  $Y_1$  graphing menu on your calculator. You can use your calculator for simple calculations like  $Y_1(4)$ , (see the discussion at the end of this paper) but “prove” means make an argument which does not mention the graph.

Use the intermediate value theorem again to decide whether the solution in of  $f(x) = 0$  lies in  $[4, 5]$  or  $[5, 6]$ .

Take the interval that you just decided contains the solution, split it in half, and decide again which new interval contains the solution.

Keep repeating the previous step, deciding each time whether the solution lies in the left or right half of the latest interval. Stop the first time that you reach an interval with length  $\leq .01$ . Take the midpoint of this interval, state this as your approximate solution, and indicate why the real solution equals the one you have just given,  $\pm .005$ .

Imagine that you were telling someone how to repeat the process you’ve just done. They don’t know about this example, but are able to follow clear, detailed instructions and do the math flawlessly (Actually, I’m thinking that this person is a computer!) Give them a step by step set of detailed directions that could be applied to any function  $f(x)$  which is defined on any interval  $[a, b]$ .

Apply the description you've just given to find the root  $g(x) = 0$  between  $x = 5$  and  $x = 7$  (with error at most .005) where  $g(x)$  is defined as

$$g(x) = \text{fnInt}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}, x, 1.5, x) \quad (2)$$

(The command `fnInt(` is found under `MATH`, then choice `9:fnInt(`. The factorial function `!` is found under `MATH`, go *right* to the menu `PRB`, then choice `4:!`.)

Now, there are a few other ways you could have calculated the solutions of  $f(x) = 0$  and  $g(x) = 0$ . One way is Newton's method, which we'll describe below. You also could have graphed these functions. How long does it take for your calculator to graph  $f(x)$  (the same  $f(x)$  as in Equation (1)) on a standard window? How long does it take for your calculator to graph  $g(x)$  (the same  $g(x)$  as in Equation (2)) on a standard window? If all you did was graph, zoom, and trace (not using any other built in functions of your calculators), how many times would you have to graph  $f(x)$  to find the solution of  $f(x) = 0$  with an accuracy of .005? How long would it take to graph  $g(x)$  this many times?

Here's a calculator program which solves for roots using the method of bisection. Note: it assumes that the function has been entered into  $Y_1$  in the graphing menu.

```
:Input "L BOUND ",L
:Input "R BOUND ",R
:Input "ACCURACY ",A
:If Y1(L)*Y1(R)>0
:Disp "SIGN ERR"
:While R-L≥2*A
:(L+R)/2→M
:If Y1(L)*Y1(M)≥0
:Then
:M→L
:Else
:M→R
:End
:End
:Disp M
```

Try entering the program and running it with  $Y_1 = f(x)$ ,  $L = 4$ ,  $R = 6$  and  $A = .005$ . How long did it take? Try running it with  $Y_1 = g(x)$ ,  $L = 5$ ,  $R = 7$  and  $A = .005$ . How long did it take?

Now we learn another method of solving for roots. It's called **Newton's method**. This method gives a sequence of approximate solutions  $x_0, x_1, x_2$ , etc. where the solutions get more and more accurate (most of the time). The  $x$ -value  $x_0$  is an initial guess, provided by you, which should be somewhere close to the real solution. After that, you use Newton's method to calculate  $x_1$ , which should come out even closer to the real solution. Then you use Newton's method again to calculate  $x_2$ , which should be even closer to the solution, etc.

Here's the definition of how to calculate  $x_1, x_2, x_3$ , etc., starting with a person supplying the

initial guess  $x_0$ , and where we are trying to solve an equation  $f(x) = 0$ .

$$\begin{aligned} x_0 &= \text{initial guess} \\ x_1 &= x_0 - f(x_0)/f'(x_0) \\ x_2 &= x_1 - f(x_1)/f'(x_1) \\ x_3 &= x_2 - f(x_2)/f'(x_2) \\ &\dots \\ x_n &= x_{n-1} - f(x_{n-1})/f'(x_{n-1}) \end{aligned}$$

For instance, suppose we wanted to use this to calculate the square root of 115. We would let the function be  $r(x) = x^2 - 115$ . Then  $r'(x) = 2x$ . Let our initial guess be  $x_0 = 10$  (which is kind of close to  $\sqrt{115}$ ). Then the calculations would go like this

$$\begin{array}{l} \text{approximations of } \sqrt{115} : \\ x_0 = 10 \\ x_1 = 10 - \frac{10^2 - 115}{2 \cdot 10} = 10.75 \\ x_2 = 10.75 - \frac{10.75^2 - 115}{2 \cdot 10.75} = 10.72383721 \\ x_3 = 10.72383721 - \frac{10.72383721^2 - 115}{2 \cdot 10.72383721} = 10.72380530 \end{array}$$

This is already a pretty good approximation of  $\sqrt{115}$  (my calculator gives this square root as 10.72380529).

Use  $f(x)$  as in Equation (1), start with  $x_0 = 4$ , and find  $x_1, x_2, x_3$ , etc. until you get two numbers  $x_i, x_{i+1}$ , that are less than .005 apart, i.e.  $|x_i - x_{i+1}| < .005$ . Show your work in the same way we showed our calculations for  $\sqrt{115}$ .

You may be asking yourself why Newton's method works. It turns out that it has a simple geometrical explanation. Suppose you pick a point  $(x_0, f(x_0))$  on the graph of a function  $f(x)$ , near where  $f(x)$  crosses the  $x$ -axis, and you draw the tangent line at  $x_0$ . Then this tangent line will cross the  $x$ -axis close to where  $f(x)$  crosses the  $x$ -axis, even closer than  $x_0$  was. The point where the tangent line crosses equals  $x_1 = x_0 - f(x_0)/f'(x_0)$ . Now you just repeat the whole process: draw a tangent line at  $x_1$ , see where this line intersects the  $x$ -axis, call that point  $x_2$ , etc.

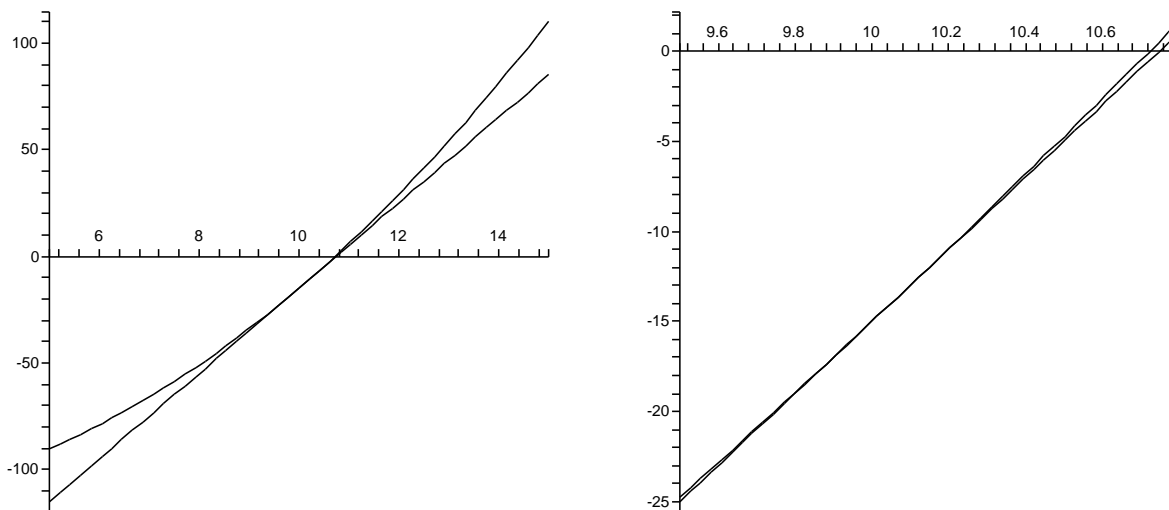
For example, when we approximated  $\sqrt{115}$ , our first point was  $x_0 = 10$ . The graph of  $f(x) = x^2 - 115$ , the tangent line at  $x_0 = 10$ , and the  $x$ -intercepts are shown in Figure 1. In the first graph it is easy to tell the difference between the curve  $f(x)$  and the straight line, which is the tangent line at  $x = 10$ , however it is not easy to see that they intersect the  $x$ -axis at different points. In the second graph, we can see that the curve and the straight line do intersect the  $x$ -axis at different points, but the  $x$ -intercepts are quite close together, which is the whole point.

Draw a large graph showing  $f(x)$  (the function in Equation (1)), and the tangent line at  $x_0 = 4$ . Label also the  $x$ -intercept of this tangent line and indicate that it equals the  $x_1$  that you found above.

Here we give the calculator program for Newton's method. The program assumes that the user will have entered the function  $f(x)$  into the  $Y_1$  graphing menu.

```
:Input "FIRST X ",X
:Input "ACCURACY ",A
:X+10→L
```

Figure 1: Newton's method for  $x^2 - 115$



```

:While abs(X-L)>A
:X→L
:X-Y1/nDeriv(Y1,X,X)→X
:Disp X
:End

```

Now show the output from your calculator program starting with  $x_0 = 6$  and accuracy  $A = .005$ .

Now, suppose you again apply Newton's method to  $f(x)$ , but this time starting with  $x_0 = 0$ . What happens? What does your calculator program do? Show a graph and explain.

Now try your calculator program on  $g(x) = 0$ , starting with  $x = 5$ , and  $A = .005$ . How long does this take to run?

Write up everything you have learned here, define the IMV bisection method, Newton's method, compare and contrast the two. Compare and contrast these methods with graphing and tracing on your calculator. Indicate which you think is better. Is it possible to combine the best parts of both approaches? Include everything that you have been asked to do up in this document. Include an introduction and a conclusion. Label your graphs, make them neat and clear.

Using your calculator to calculate  $Y_1(4)$  and things like this.

First, you need to enter a function into  $Y_1$ . Once you do this, you can calculate  $Y_1(4)$  on the graphing menu, or directly on the main menu.

To enter  $Y_1(4)$  on the main menu, hit **VARs**, go right to the menu **Y-VARS**, choose **1:Function**, then choose **Y<sub>1</sub>**. Now enter the parentheses ( ) and the number 4.

Finally, if you are entering lots of numbers like this, there's a way to save a bunch of steps so that you don't have to hit this many buttons. On the main menu, you can have the calculator recall the last thing you entered by hitting **2ND**, **ENTER**. If you repeat this, you can have the calculator recall what you entered two times ago, or repeat it again and get what was entered three times ago, etc.