1. I will ask you to prove two of the following:
   (proofs in notes and/or book)

2. Differentiate the following functions.
   
   (a) \( f(x) = \frac{x}{1+x} \).

   Use the Quotient Rule:
   
   \[
   f'(x) = \frac{1 \cdot (1 + x) - x \cdot 1}{(1 + x)^2}
   = \frac{1 + x - x}{(1 + x)^2}
   = \frac{1}{(1 + x)^2}
   \]

   (b) \( g(x) = \pi x^e - e^\pi x \).

   Use the general Power Rule:
   
   \[
   g'(x) = \pi e x^{e-1} - e^\pi
   \]

   (c) \( h(x) = x \cdot e^x \).

   Use the Product Rule:
   
   \[
   h'(x) = f'g + fg'
   = 1 \cdot e^x + x \cdot e^x
   = e^x + xe^x
   \]

   (d) \( l(x) = \frac{e^x + 3\sqrt{x}}{2x^2 + 5} \).

   Use the Quotient Rule:
   
   \[
   f = e^x + 3\sqrt{x} = e^x + 3x^{1/2}, \quad g = 2x^2 + 5
   \]

   \[
   \Rightarrow f' = e^x + \frac{3}{2}x^{-1/2}, \quad g' = 4x
   \]

   \[
   l'(x) = \frac{f'g - fg'}{g^2}
   = \frac{(e^x + \frac{3}{2}x^{-1/2})(2x^2 + 5) - (e^x + 3\sqrt{x})(4x)}{(2x^2 + 5)^2}
   \]

   At this point, simplify at your own risk!
(e) \( y = (1 - x^{-1})^{-2} \)

Use the Chain Rule:

\[
\begin{align*}
y &= f \circ g \\
f(x) &= x^{-2} \quad g(x) = 1 - x^{-1} \\
f'(x) &= -2x^{-3} \\
g'(x) &= -2(1 - x^{-1})^{-3} \\
y' &= f'(g(x))g'(x) \\
&= -2(1 - x^{-1})^{-3}(x^{-2})
\end{align*}
\]

OR, using Leibniz notation,

\[
\begin{align*}
y &= u^{-2} \quad u = 1 - x^{-1} \\
dy &= -2u^{-3} \quad du = x^{-2} \\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= -2u^{-3}x^{-2} \\
&= -2(1 - x^{-1})^{-3}(x^{-2})
\end{align*}
\]

(f) \( y = \cos(\tan x) \)

Chain Rule: \( y = \cos u, \quad u = \tan x \)

\[
\begin{align*}
y' &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= (-\sin u)(\sec^2 x) \\
&= (-\sin(\tan x))(\sec^2 x)
\end{align*}
\]

OR, using prime notation

\[
\begin{align*}
y &= f \circ g \\
f(x) &= \cos x \\
g(x) &= \tan x \\
f'(x) &= -\sin x \\
g'(x) &= \sec^2 x \\
y' &= f'(g(x))g'(x) \\
&= -\sin(\tan x)\sec^2 x
\end{align*}
\]

(g) \( f(x) = (x + 2)^4(11x + 1)^2 \)

Chain Rule and Product Rule combined:

\[
\begin{align*}
f'(x) &= [4(x + 2)^3(1)](11x + 1)^2 + [(x + 2)^4][2(11x + 1)^1(11)] \\
&= 4(x + 2)^3(11x + 1)^2 + 22(x + 2)^4(11x + 1)
\end{align*}
\]

(h) \( g(x) = \cot(3x^2 - x + 5) \)

Chain rule: \( y = \cot u, \quad u = 3x^2 - x + 5 \)

\[
\begin{align*}
g'(x) &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= -\csc^2 u(6x - 1) \\
&= -\csc^2(3x^2 - x + 5)(6x - 1)
\end{align*}
\]

(i) \( h(x) = \frac{\tan x}{e^x} \)
Quotient Rule:

\[ h'(x) = \frac{\sec^2 x \cdot e^x - \tan x \cdot e^x}{e^{2x}} = \frac{\sec^2 x - \tan x}{e^x} \]

OR Product Rule:

\[ h(x) = \tan x \cdot e^{-x} \]
\[ h'(x) = \sec^2 x \cdot e^{-x} + \tan x(-e^{-x}) = \frac{\sec^2 x \cdot e^{-x} - \tan x \cdot e^{-x}}{e^x} \]

(j) \[ y = e^{4x^2} \]

\[ y' = 8xe^{4x^2} \]

(k) \[ y = \frac{e^{2x}}{1 + \sec x} \]

\[ y' = \frac{2e^{2x}(1 + \sec x) - e^{2x}(sec x \tan x)}{(1 + \sec x)^2} \]

simplify at own risk!

(l) \[ y = e^{ex} \]

\[ y' = e^x e^x \]

(m) \[ y = \ln \sqrt{x^3 + x - 1} \]

Use the Chain Rule w/ log functions:

\[ y' = \frac{1}{\sqrt{x^3 + x - 1}} \left( \frac{1}{2} (x^3 + x - 1)^{-1/2} (3x^2 + 1) \right) \]
\[ = \frac{3x^2 + 1}{2\sqrt{x^3 + x + 1}(x^3 + x - 1)} \]
(n) \( y = \tan^{-1}(9x^2) \)

Use the Chain Rule w/ inverse trig functions:

\[
y = \tan^{-1}(f(x)) \\
\Rightarrow y' = \frac{f'(x)}{1 + (f(x))^2} \\
\Rightarrow y' = \frac{18x}{1 + 81x^4}
\]

(o) \( y = \sqrt[3]{\sin^{-1}x} \)

\[
y = (\sin^{-1}x)^{\frac{1}{3}} \Rightarrow y' = \frac{1}{3}(\sin^{-1}x)^{-\frac{2}{3}}(\sin^{-1}x)' = \frac{1}{3(\sin^{-1}x)^{2/3}} \left( \frac{1}{\sqrt{1 - x^2}} \right)
\]

(p) \( y = 3^x \cos x \)

Use the chain rule w/ exp functions: \( y' = 3^x \cos x \ln 3(x \cos x)' \), then use the product rule on \( x \cos x \) to get

\[
y' = 3^x \cos x \ln 3(\cos x - x \sin x)
\]

(q) \( y = \frac{e^{2x}}{1 + \sec x} \)

Use the quotient rule: \( y' = \frac{(e^{2x})(1 + \sec x) - e^{2x}(1 + \sec x)'}{(1 + \sec x)^2} \)

\[
= \frac{(e^{2x} \cdot 2)(1 + \sec x) - e^{2x}(\sec x \tan x)}{(1 + \sec x)^2} \\
= \frac{2e^{2x}(1 + \sec x) - e^{2x}(\sec x \tan x)}{(1 + \sec x)^2}
\]

simplify at own risk from here!

(r) \( y = \sqrt{\ln x} \)

\[
y = (\ln x)^{\frac{1}{2}} \Rightarrow y' = \frac{1}{2} (\ln x)^{-\frac{1}{2}} (\ln x)' = \frac{1}{2\sqrt{\ln x}} \left( \frac{1}{x} \right) = \frac{1}{2x\sqrt{\ln x}}
\]

(s) \( y = \log_3(23x - 4x^2 + e^2) \)

Use chain rule with log. functions: \( y' = \frac{(23x - 4x^2 + e^2)'}{\ln 3(23x - 4x^2 + e^2)} = \frac{23 - 8x + 2xe^2}{\ln 3(23x - 4x^2 + e^2)} \)
(t) \( y = (5x^2 - 9x) \ln x \)

\[
y' = (5x^2 - 9x)' \ln x + (5x^2 - 9x)(\ln x)' = (10x - 9) \ln x + \frac{5x^2 - 9x}{x}
\]

(u) \( y = \sin^{-1}(\ln x) \)

Use the chain rule w/ inverse trig functions:

\[
(sin^{-1} f(x))' = \frac{1}{\sqrt{1 - (f(x))^2}} \cdot f'(x)
\]

\[
= \frac{1}{\sqrt{1 - (\ln x)^2}} \cdot \frac{1}{x}
\]

(v) \( y = \cos(\ln x) - e^3 \)

\[
y' = -\frac{\sin(\ln x)}{x}
\]

(w) \( f(x) = (\cos x)^{\sin x} \)

Use logarithmic differentiation:

\[
y = (\cos x)^{\sin x} \implies \ln y = \ln(\cos x)^{\sin x} = \sin x \cdot \ln(\cos x)
\]

\[
\frac{1}{y} \cdot y' = \cos x \cdot \ln(\cos x) + \sin x \cdot -\frac{\sin x}{\cos x}
\]

\[
y' = y \left[ \cos x \cdot \ln(\cos x) - \sin x \tan x \right]
\]

\[
f'(x) = (\cos x)^{\sin x} \left[ \cos x \cdot \ln(\cos x) - \sin x \tan x \right]
\]

(x) \( g(x) = (x^2 + 1)^x \)

Use logarithmic differentiation:

\[
y = (x^2 + 1)^x \implies \ln y = \ln(x^2 + 1)^x = x \cdot \ln(x^2 + 1)
\]

\[
\frac{1}{y} \cdot y' = 1 \cdot \ln(x^2 + 1) + x \cdot \frac{2x}{x^2 + 1}
\]

\[
y' = y \left[ \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right]
\]

\[
g'(x) = (x^2 + 1)^x \left[ \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right]
\]
3. Find \( y'' \) for the following functions.

(a) \( y = 3x^9 - 7x^6 - 3x + 1 \)

\[
y' = 27x^8 - 42x^5 - 3
\]

\[
y'' = 216x^7 - 210x^4
\]

(b) \( y = \tan 3x \)

\[
y' = 3\sec^2(3x) = 3(\sec 3x)^2
\]

\[
y'' = 6(\sec 3x)(\sec 3x \tan 3x) = 18\sec^2 3x \tan 3x
\]

(c) \( y = xe^{5x} \)

\[
y' = e^{5x} + 5xe^{5x}
\]

\[
y'' = 5e^{5x} + 5e^{5x} + 25xe^{5x} = 10e^{5x} + 25xe^{5x}
\]

4. Find \( h' \) in terms of \( f' \) and \( g' \).

(a) \( h(x) = \frac{f(x)g(x)}{f(x) + 2g(x)} \)

\[
h'(x) = \frac{(fg)'(f + 2g) - (fg)(f + 2g)'}{(f + 2g)^2}
\]

\[
= \frac{(f'g + fg')(f + 2g) - (fg)(f' + 2g')}{(f + 2g)^2}
\]

\[
= \frac{f'g(f + 2g) + f^2g' + 2fg'g - fgf' - 2fgg'}{(f + 2g)^2}
\]

\[
= \frac{2f'g^2 + f^2g'}{(f + 2g)^2}
\]

Simplify at your own risk!

(b) \( h(x) = f(g(\sin 4x)) \)

\[
h'(x) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot 4\cos 4x
\]

\[
= 4\cos 4x f'(g(\sin 4x))g'(\sin 4x)
\]

(c) \( h(x) = f(x)e^{g(x)} \)

\[
h'(x) = f'e^{g} + fe^{g}g'
\]

\[
= f'(x)e^{g(x)} + f(x)g'(x)e^{g(x)}
\]
5. Find the equation of the line tangent to the curve $\sqrt{x+2} + \sqrt{y} = 3$ at the point $(2, 1)$.

Use implicit differentiation on $(x + 2)^{\frac{1}{2}} + y^{\frac{1}{2}} = 3$:

$$\frac{1}{2}(x + 2)^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = -\frac{1}{2\sqrt{x+2}}$$

$$\frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x+2}}$$

$$= -\frac{\sqrt{y}}{\sqrt{x+2}}$$

At the point $(2, 1)$, $\frac{dy}{dx} = -\frac{\sqrt{1}}{\sqrt{2+2}} = -\frac{1}{2}$

So, using point-slope form, $y = -\frac{1}{2} \cdot 2 + b \Rightarrow 1 = -1 + b \Rightarrow b = 2$

$$y = -\frac{1}{2}x + 2$$

6. Find the equation of the line tangent to the curve $x^2 - 2xy + y^3 = 2 + xe^y$ at the point $(-1, 0)$.

$$2x - (2y + 2xy') + 3y^2y' = e^y + xe^y$$

$$3y^2y' - 2xy' - xe^yy' = e^y + 2y - 2x$$

$$y'(3y^2 - 2x - xe^y) = e^y + 2y - 2x$$

$$y' = \frac{e^y + 2y - 2x}{3y^2 - 2x - xe^y}$$

$$m = \frac{dy}{dx} \text{ or } y' \text{ when } x = -1, y = 0$$

$$= \frac{1 + 0 + 2}{0 + 2 + 1} = 1$$

$$0 = 1(-1) + b \Rightarrow b = 1$$

$$y = x + 1$$
7. Using exact values, find the equation tangent to \( f(x) = \sin^2 x \) at \( x = \frac{\pi}{3} \).

\[
f'(x) = 2 \sin x \cos x \implies m = f'(\frac{\pi}{3}) = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}
\]

\[
f\left(\frac{\pi}{3}\right) = \left(\sin\frac{\pi}{3}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}
\]

\[
\frac{3}{4} = \frac{\sqrt{3}}{2} \cdot \frac{\pi}{3} + b \implies b = \frac{3}{4} - \frac{\pi \sqrt{3}}{6}
\]

\[
y = \frac{\sqrt{3}}{2} x + \frac{3}{4} - \frac{\pi \sqrt{3}}{6}
\]

8. Find \( \frac{dy}{dx} \) or \( y' \) by implicit differentiation:

(a) \( 3x^2 - 5y^3 = x \)

\[
6x - 15y^2 y' = 1
\]

\[
15y^2 y' = 6x - 1
\]

\[
y' = \frac{6x - 1}{15y^2}
\]

(b) \( 5xy + \cos x = e^y \).

\[
5y + 5xy' - \sin x = e^y \cdot y'
\]

\[
5xy' - e^y \cdot y' = \sin x - 5y
\]

\[
y'(5x - e^y) = \sin x - 5y
\]

\[
y' = \frac{\sin x - 5y}{5x - e^y}
\]

9. Using exact values, find the equation tangent to \( g(x) = \cos(\pi x) + 3 \) at

(a) \( x = -2 \)

\[
g'(x) = -\pi \sin(\pi x) \implies m = g'(-2) = 0 \text{ (so horiz tangent line)}
\]

\[
y = g(-2) = 1 + 3 = 4
\]

\[
y = 4
\]
(b) \( x = \frac{1}{2} \)

\[ g'(x) = -\pi \sin(\pi x) \implies m = g'\left(\frac{1}{2}\right) = -\pi \sin\left(\frac{\pi}{2}\right) = -\pi \]

\[ g\left(\frac{1}{2}\right) = \cos\left(\frac{\pi}{2}\right) + 3 = 3 \]

\[ 3 = -\pi \cdot \frac{1}{2} + b \implies b = \frac{\pi}{2} + 3 \]

\[ y = -\pi x + \frac{\pi}{2} + 3 \]

10. The volume of a right cylinder is given by the equation \( V = \pi r^2 h \), where \( r \) is the radius of the top and base and \( h \) is its height.

(a) Find the rate of change of the volume with respect to the height if the radius is constant.

\[ \frac{dV}{dh} = \pi r^2 \]

(b) Find the rate of change of the volume with respect to the radius if the height is constant.

\[ \frac{dV}{dr} = 2\pi rh \]

11. A particle moves on a horizontal line so that at time \( t \) seconds, its position along the line is given by \( s(t) = t^3 - 12t + 3 \), where position is measured in meters (\( t \geq 0 \)).

(a) Find the velocity and acceleration functions.

\[ v = s'(t) = 3t^2 - 12 \]

\[ a = v' = s'' = 6t \]

\[ v(t) = 3t^2 - 12, \quad a(t) = 6t \]

(b) When is the particle at rest?

particle at rest when \( v = 0 \) so solving for \( v = 0 \)

\[ 3t^2 - 12 = 0 \]

\[ t^2 = 4 \]

\[ t = \pm 2 \quad t = -2 \text{ doesn’t make sense here} \]

\[ 2 \text{ seconds} \]

(c) What is the particle’s velocity at 3 seconds? What is its acceleration at this time?

\[ v(3) = 3 \cdot 9 - 12 = 15, \quad a(3) = 18 \]

velocity is 15 m/s, acceleration is 18 m/s^2
(d) When is the particle moving to the right (positive velocity)? When is the particle moving to the left?

Want $v > 0$, so find when $v = 0$ and look on either side – already found $v = 0$ when $t = 2$. When $0 < t < 2$, $3t^2 - 12 < 0$ and when $t > 2$, $3t^2 - 12 > 0$.

The particle is moving to the right after 2 seconds and the particle is moving to the left between 0 and 2 seconds.

(e) When is the particle speeding up? When is it slowing down?

The particle is speeding up when velocity and acceleration have the same sign and the particle is slowing down when velocity and acceleration when $v$ and $a$ have opposite signs. Since $a = 6t$, $a$ is positive for $t > 0$. And by above, $v > 0$ when $t > 2$ and $v < 0$ when $t \in (0, 2)$.

The particle is speeding up after 2 seconds and slowing down between 0 and 2 seconds.

12. The equation of motion of a particle is given as $s = t^4 - 4t^3 + 2$ where $s$ is in meters and $t$ is in seconds.

(a) Find the velocity and acceleration functions.

$v(t) = s'(t) = 4t^3 - 12t^2$  \hspace{1cm}  $a(t) = v'(t) = s''(t) = 12t^2 - 24t$

(b) Find the time(s) at which the acceleration is 0.

$12t^2 - 24t = 0 \implies 12t(t - 2) = 0$

$t = 0, 2$ seconds

(c) Find the position of the particle and the velocity at the time(s) from part (b).

Position: $t = 0 : \quad s(0) = 2$ m \hspace{1cm} $t = 2 : \quad s(2) = -14$ m

Velocity: $t = 0 : \quad v(0) = 0$ m/s \hspace{1cm} $t = 2 : \quad v(2) = -16$ m/s

13. A particle moves along the curve $y = \sqrt{1 + x^3}$. As it reaches the point $(2, 3)$, the $y$-coordinate is increasing at a rate of 4 cm/s. How fast is the $x$-coordinate of the point changing at that instant?

$$y = (1 + x^3)^{1/2}$$

$$\frac{dx}{dt} = ? \quad \text{when} \quad x = 2, \quad y = 3, \quad \text{and} \quad \frac{dy}{dt} = 4$$

$$\frac{dy}{dt} = \frac{1}{2} \cdot (1 + x^3)^{-\frac{1}{2}} \cdot (3x^2) \cdot \frac{dx}{dt}$$

$$= \frac{3x^2}{2\sqrt{1 + x^3}} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{2\sqrt{1 + x^3}}{3x^2} \cdot \frac{dy}{dt}$$

$$\frac{dx}{dt} \Big|_{x=2, \ y=3} = \frac{2\sqrt{1 + 8}}{3 \cdot 4} \cdot 4 = 2 \text{ cm/s}$$
14. A baseball diamond is a square with side 90 feet. A batter hits the ball and runs toward first base with a speed of 24 ft/s.

(a) At what rate is his distance from second base decreasing when he is halfway to first base?

If \( x \) is the distance between the batter and first base, then \( x \) is decreasing at a rate of 24 ft/s so \( \frac{dx}{dt} = -24 \). Let \( y \) be the distance between the batter and second base. Then, by the Pythagorean Thm, we have \( y^2 = x^2 + 90^2 \). What is \( \frac{dy}{dt} \) when \( x = 45 \) and thus \( y = \sqrt{10125} \)?

\[
2y \cdot \frac{dy}{dt} = 2x \cdot \frac{dx}{dt} \implies \frac{dy}{dt} = \frac{2(45)(-24)}{2\sqrt{10125}} = -\frac{1080}{\sqrt{10125}} \text{ so decreasing at about 10.7 ft }/s
\]

(b) At what rate is his distance from third base increasing at the same moment?

If \( x \) is the distance between the batter and third base, then \( x \) is increasing at a rate of 24 ft/s so \( \frac{dx}{dt} = 24 \). Let \( y \) be the distance between the batter and third base. Then, by the Pythagorean Thm, we have \( y^2 = x^2 + 90^2 \). What is \( \frac{dy}{dt} \) when \( x = 45 \) and thus \( y = \sqrt{10125} \)?

The only difference between (b) and (a) in the computations is \( \frac{dx}{dt} \) so we have

\[
\frac{dy}{dt} = \frac{1080}{\sqrt{10125}} \text{ so increasing at about 10.7 ft }/s
\]

15. The volume of a cube is increasing at a rate of 10 cm\(^3\)/min. How fast is the surface area increasing when the length of an edge is 30 cm?

\[
V = x^3 \implies \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}
\]

\[
SA = 6x^2 \implies \frac{dSA}{dt} = 12x \cdot \frac{dx}{dt}
\]

\[
\frac{dSA}{dt} =? \text{ when } \frac{dV}{dt} = 10 \text{ and } x = 30
\]

\[
10 = 3(30)^2 \cdot \frac{dx}{dt} \implies \frac{dx}{dt} = \frac{1}{270}
\]

\[
\frac{dSA}{dt} = 12(30) \cdot \frac{1}{270} = \frac{4}{3} \text{ cm}^2/\text{min}
\]
16. The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is \( \pi/6? \)

\[
\cot \theta = \frac{x}{400} \\
- \csc^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{400} \cdot \frac{dx}{dt} \Rightarrow \\
-400 \cdot \csc^2 \theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt} \\
\frac{d\theta}{dt} = -0.25 \text{ rad/h.}
\]

When \( \theta = \pi/6, \csc^2 \theta = 4 \Rightarrow \frac{dx}{dt} = -400 \cdot 4 \cdot (-0.25) = \boxed{400 \text{ ft/hr.}} \]